专题论文集

SELECTED PAPERS OF O.C.ZIENKIEWICZ ON FINITE ELEMENT METHOD

辛克维支有限元法论文选集

编者 钱令希

中国学术出版社

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选编者的前言

有限元法，今天已成为解决连续体介质力学的强有力工具。人们不仅用它解决了许多过去难以下手的问题，同时也看到了它还有很大的潜在力量，可以进一步发挥。只要问题的物理方面，诸如介质的本构关系、运动的方程、现象的本质等等弄清楚以后，通过有限元法总可以得到问题的数值解答。这在电子计算机和有限元法问世以前是做不到的。有限元法的成功，使人们对它寄予了更多的期望。期望和实践，促使有限元法研究与领域的不断深化和扩大。

这本选集的作者对有限元法的贡献是众所公认的。为此，他得到过许多荣誉，包括英国皇家学会的会员和美国工程科学院的国外院士等称号。他在英国斯旺西的威尔士大学领导着一支教学和科研的梯队，卓有成效地进行研究工作。本集是从250多篇论文中选择出来的。 作者的工作面很宽，成果非常多，而选集的篇幅有限，所以编选是相当困难的。我们选择的目标，主要着眼于发挥有限元法潜力和扩大应用领域方面的工作，这是当前我们最感兴趣的。我们也注意了作者在应用中发展有限元法本身的理论和方法。

本书的内容可以作如下分类：

(1)概论 1 篇、(2)理论与方法 5 篇、(3)桥梁 2 篇、(4)房屋 1 篇、(5)拱坝 2 篇、(6)结构振动 1 篇、(7)透平叶片 1 篇、(8)场的问题 4 篇、(9)非线性问题 6 篇、(10)金属挤压与成型 2 篇、(11)土、岩力学 6 篇、(12)近海平台地基 2 篇、(13)钢筋混凝土结构 1 篇、(14)流体、固体相互作用 4 篇、(15)波浪问题 2 篇、(16)热问题 2 篇、(17)结构型式优化 1 篇。

辛克维支教授是学土木工程出身的。1943年在英国皇家学院毕业后，他跟索思韦尔教授做过有限差分法和松弛法的研究，又从事水坝工程的实际设计和建造。作为一个力学家，他有很强的工程观点；作为一个工程师，他又有很深的数学和力学修养。所以他在六十年代初接触有限元法之后，立即洞察到这个方法的理论基础和实际意义。他运用自如，步步扩大应用的领域，并结合具体情况研究经济和有效的方法。他在解决许多线性固体力学问题的同时，发现有限元法可以解决广泛的各种连续场问题，如流体、温度和电磁场等问题。接着他又深入到非线性和有时间效应的动力问题中去。他善于抓问题的实质，简化计算的模型和找出高效率的计算方法。例如在难度很大的非线性领域中，对材料非线性问题、从最简单的理想塑性、到粘性流、蠕变现象，直至像土壤和岩石这样极为复杂的性质，他都能运用工程观点、力学概念和恰当的有限元法得到数值解答。

希望读者们在各自感兴趣的方面，从这本选集中能得到收获。唐立民教授和我在本集的选编中付出了不少劳动，同时我们也从中得到了相当的收获。

钱令希

1982年 2 月于大连工学院

PREFACE

It is indeed a great honour for any scientific worker to see his contributions published in a collected form. Such a publication provides a record of the history of his own development and contributions he and his colleagues have made to the subject. It remains thus a memento of a personal kind for his family and successors as well as recording small steps taken in progressing and pushing forward the frontiers of his subject.

The selection of the papers was made by Professor Qian and his colleagues and covers a twenty year period of work at University College of Swansea in which finite elements play a dominant part. My entry into the subject area was much motivated by contact with Prof. Clough of Berkeley who first introduced me to the fascinating field of finite elements. However, the fundamental seeds of interest in a numerical solution of engineering problems came earlier. Here, I would like to mention Prof. A. J. S. Pippard and Prof. R. B. Southwell who respectively gave me an interest in application of theory to the structural field and showed me the power of numerical calculations in the early days based on finite difference, calculus and relaxation methods. Much time has passed since those early days, but perhaps the reader will discern theroughout the many publications a continuing thread of attempting to unify and generalise the procedures showing their relation to history.

I hope this collection will be of use to my Chinese colleagues who in recent years have shown so much interest in the subject and contributed proportionately. I wish them continued success with their efforts.

O. C. Zienkiewicz

SELECTED PAPERS OF O.C. ZIENKIEWICZ

ON FINITE ELEMENT METHOD

GENERAL

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"The Finite Element Method: from Intuition to Generality"
App. Mechs. Review, 249-256, March 1970.

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"Optimization of Shape to Minimize Stress Concentration"
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The Finite Element Method: From Intuition to Generality

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Introduction

The engineer in need of numbers with which his design process can be described is often impatient with the "properly formulated" problem for which complex equations exist but for which only comparatively trivial solutions can be achieved by classical mathematics. At such times, he often formulates ad hoc, crude, models which, having served the purpose once, are later discarded. On some occasions, however, his inelegant approach leads to a discovery (or rediscovery) of powerful and fundamental methods. Examples ranging from Heaviside's operator techniques to Southwell's demonstration of practical use of finite differences can be quoted. The finite element method must now be viewed in the same category. Its popularity among engineers is assured and, at long last, an arousal of interest among applied mathematicians promises, at least, a consolidation of fundamentals if not more rapid progress.

The methods of dealing with discrete structural systems or indeed with similar hydraulic and electrical networks are well understood by the practitioner. Many computer programs are available here and indeed were among the earliest to be developed after the advent of this "electronic slave." In perhaps a naive, but essentially physical, manner, attempts to reduce a continuum to an assembly of *discrete,* equivalent, elements frequently have been made. Here, in the context of an elastic continuum, the first attempts tried to assimilate the continuum behavior into an assembly of rods or beams which were standard ingredients of the engineer's tool kit. The early work of Hrenikoff [Ref. 1] in 1941 McHenry [Ref. 2] in 1943, and Newmark [Ref. 3] in 1949 should be noted. However, the credit of approximating directly to a continuum region by an element with multiple connecting points must go to Turner, et al., [Ref. 4] in 1956 and Clough [Ref. 5] in 1960 and Argyris [Ref. 6], 1955. Here for the first time, the magic name of "finite element" is used, but the derivation of the basic properties is achieved by physical arguments relating to stress or displacement distribution within the subregion. The fact that at that time the computer begins to be an effective tool leads, however, rapidly to the possibility of solving real complex elastic solutions.

In attempts to tidy up the basis of deriving stiffness properties of elements, it is rapidly realized that *one way* of doing this is to assume a displacement pattern defined in terms of nodal displacements, element by element, so that continuity is observed and internal forces are obtained by virtual work. The identity of this process with that of approximate minimization of total potential energy similar to Rayleigh-Ritz processes becomes obvious [Ref. 7]. Indeed, the main difference in the method from that previously used as a standard approximation is the *piecewise continuous field definition* enabling irregular boundaries to be simply fitted and avoiding thus the obvious limitations.

Further, the second essential merit of this piecewise definition becomes apparent. Unlike in the "standard" Ritz process, the minimization equations form *banded matrices* (or at least sparse) for which solution by direct methods or iteration can be obtained conveniently.

Two matters become clear at that stage. The first is that if the finite element process is so simply described, it may be a "rediscovery" of known mathematical methods. The second is that its range could be extended to other situations where (quadratic) functionals have to minimized. The first point is rapidly answered by retrieval of the now classic work by Courant in 1943 [Ref. 8] in which the essence of a triangular finite element is formulated and of the parallel work of Prager and Synge [Ref. 9] which led to the "hypercicle" method [Refs. 10, 11].

The second point is followed by a rapid extension of the finite element process into nonstructural fields by Zienkiewicz [Ref. 12] and others [Refs. 13-16] who demonstrate the applicability to fluid mechanics, heat conduction and indeed other problems governed by the quasi-harmonic differential forms.

Further, if the variational process allows a finite element formulation then (a) other parameters than nodal unknown values could be included in the analysis, and (b) many alternative formulations of same problems are possible.

Pian [Ref. 17] shows how, in the context of elasticity, nodeless parameters can be introduced and indeed treated in otherwise standard manner. The retention of some of the nodal parameters is, however, always essential to preserve required interelement continuities.

Again in the context of elastic analysis it is shown that different variational forms can be used. Veubeke [Ref. 18, 19] demonstrates the use of equilibrating stress distributions and use of complementary potential energy as the functional to be minimized. (A particularly simple device for obtaining such equilibrating fields and the corresponding duals is derived by Veubeke and Zienkiewicz [Ref. 20]. Reissner functional is used with success for the first time by Herrmann [Ref. 21] while other possible hybrid formulations are discussed in detail by Pian [Ref. 22]. It is, however, once again instructive to observe that the complex hybrid functionals were derived by Pian long after the elements of that particular type have been introduced and used successfully *on the basis of physical intuition alone* [Refs. 23, 24].

Indeed, the last remark prompts the writer to observe that, even today, some of the very successful elements having apparently no correct variational form are known to converge and yield extremely accurate approximations. Is the variational form the only one for which finite elements can be properly derived? It will be shown that other alternatives are open.

Some Mathematics of Finite Elements

If a quadratic functional to be minimized over the particular domain is χ and is defined by an integral of the unknown function $\{\phi\}$ and some of its derivatives, then, if $\{\phi\}$ is described piecewise element by element in terms of coordinate, "shape" function $[N]$ and (unknown) nodal parameters $\{\bar{\phi}\}$, the set of minimizing equations takes the form [Ref. 25]

$$\frac{\partial \chi}{\partial \{\bar{\phi}\}} = \begin{Bmatrix} \frac{\partial \chi}{\partial \bar{\phi}_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \chi}{\partial \bar{\phi}_2} \end{Bmatrix} = [K]\{\bar{\phi}\} + \{F\} = 0. \qquad (1)$$

With

$$\{\phi\} = [N]\{\bar{\phi}\}^e, \qquad (2)$$

for each element, the contributions of each element, χ^e are found to be

$$\frac{\partial \{\chi\}^e}{\partial \{\bar{\phi}\}^e} = [k^e]\{\bar{\phi}\}^e + \{F^e\} \qquad (3)$$

and *the important network topological assembly rules apply*, i.e.,

$$[K_{ij}] = \sum_{e=1}^{m} [k_{ij}^e] \text{ and } \{F_i\} = \sum_{e=1}^{m} [F_i^e]. \qquad (4)$$

The similarity with the standard structural problem is evident whatever the physical or mathematical nature of the functional.

The formulation is almost automatic once the coordinate, shape functions $[N]$ have been defined.

If the functional involved is quadratic, the matrices $[K]$ and $\{F\}$ and the similar contributions of elements are independent of the nodal parameters and can be evaluated explicitly.

Definition of such shape functions is not, however, completely arbitrary. Certain completeness criteria have to be observed if convergence to the correct solution with decreasing element size is to be observed.

In the first place, on interelement boundaries certain continuity requirements must be observed so that no contribution to χ arises from there [Refs. 7, 25]. Secondly, assuming no singular behavior, arbitrary constant, values of the integrand of χ have to attainable as element size decreases. In the elasticity case, this is the constant strain criterion given by Bazeley, et al., [Ref. 26]. Some generalizations of the convergence criteria which all fall within the "completeness" definition of Mikhlin [Ref. 27] are discussed by Pian [Ref. 28] Key [Ref. 29] Oliveira [Ref. 30] and Johnson [Ref. 103].

While most of finite element formulation fits within above variational category, certain elements violating the above criteria are known to converge monotonically to correct values at least for certain mesh subdivisions. For instance, a triangular plate element violating (slope) continuity requirements presented by Bazeley, et al., [Ref. 26] is shown there to be exactly convergent (for meshes generated by three sets of parallel lines), while an older and similarly nonconforming rectangle [Refs. 31, 32] is shown to converge by Walz, et al., [Ref. 33]. Another extremely successful element not based on any known variation principle has been derived by Melosh [Ref. 34]. Why such forms exhibit a generally satisfactory behavior is not yet fully understood, but perhaps there is some sense in physically identifiable modelling.

Other, more formal ways are, however, open to finite element formulation which need not be associated with a variational formulation. All *weighted residual* processes of solution of a differential equation [Ref. 35]

$$A(\{\phi\}) = 0 \qquad (5)$$

will lead to the standard assembly form once the shape of the unknown function is described by Eq. (2). Collocation by points or areas can be so represented, and indeed if the weighting function is represented by the shape function itself, then the Galerkin process is obtained. In some cases, application of such approaches will lead to precisely the same results as obtainable variationally [Refs. 35,36], but the range of application may well be extended now to problems where variational functionals do not exist or have to be artificially contrived. A typical example of this may be the application of finite element–Galerkin formulation in the time domain of transient problems [Ref. 37]. Yet other approaches to finite element formulation have been suggested by Oden [Ref. 38].

Matrices and Finite Elements

With matrix methods applied to the organization of structural calculation at about the same time as finite elements first made their appearance [Refs. 6, 39, 40] there has been some identification of the two, distinct, processes. Clearly, if matrix methods are the most efficient process at organizing the solution of the discrete, network, problem, they will be used in assembly and solution of the typical Eq. (1) resulting from discretization. Equally clearly, the process of finite element method which reduces the continuum description to that of a discrete model forms the essential approximation which has nothing to do with the solution technique. The writer is prompted to make this remark to illuminate the nonstructural applied mechanician who may well be put off by the complex ritual of so-called "matrix methods in structural mechanics."

(Further, this clarification is perhaps called for in view of the frequent confusion of *relaxation method* and *finite difference process* in a previous generation.)

In some recent texts on matrix methods finite elements are mentioned explicitly and such texts are valuable to the student wishing to acquire familiarity with the standard operation [Refs. 41, 104, 105].

Finite Element Versus Finite Differences

In the early days of the finite element process it often has been argued that little advantage over the finite difference discretization processes exists. Now, on occasion it is argued that the two processes are in fact identical. It is perhaps necessary to introduce some appropriate semantics before such arguments are meaningful. If we mean by the finite difference method a "localized, direct, approximation" to the "differential governing equation" [Ref. 42], then we can list on the credit side of the finite element process (derived in an integral manner):

(a) the ease of arbitrary positioning of nodal points (economy),
(b) the infinite possibilities of generation of "improved" elements by simply increasing the number of element parameters,
(c) the improvement in boundary-value approximation due to its integral form, and
(d) the ease with which different types and sizes of elements can be adopted.

Item (c) is of the utmost importance, and some recent publications comparing the assembled finite element and finite difference algorithms assess their relative accuracies without reference to the boundary conditions. Here, usually, standard finite difference operators introduce largest truncation error if any gradient conditions are specified.

In recent years, however, a number of "finite difference" processes have appeared in which only lower order derivatives are directly approximated and the final algorithm is assembled via the application of variational principles [Refs. 43, 44]. These clearly bridge some of the computational gaps between the two approaches, and indeed some of the obvious advantages disappear. Nevertheless points (a), (b) and (d) above are still valid.

Finally, although not directly relevant to efficiency of computation, the physical interpretation which can be given to the finite elements gives the user a sense of reality. This "psychological" point is of twofold benefit. First, gross errors of formulation are easily detected, and second, unexpected extensions of the process occur to the users.

The Importance of Shape Functions

With the type of problem and variational (or other) formulation decided, the only major step is the decision regarding the shape function form. Once this is done all remaining algebra and computation follows a standard pattern. The importance of the choice of suitable shape function is obvious.

The simplest form of space subdivision is the triangle for two, and tetrahedron for three, dimensions. Such elements with nodes placed at the vertices are among the earliest used [Refs. 4, 45] in problems for which the definition of χ needs only continuity of the unknown function (elasticity, quasi-harmonic problems, etc.).

The observation that if the total number of degrees of freedom associated with an element is increased then *equal accuracy of the assembled problem can be obtained with fewer degrees of freedom* led to the introduction of more complex elements. Triangles and tetrahedra with nodes placed at midsides were introduced by Veubeke [Ref. 18] and Argyris [Ref. 46], respectively. Today, complete families of such and other elements can be generated explicitly [Ref. 47], see Fig. 1.

With few, large elements, the merit of close boundary representation disappears unless the element can become curved. The introduction of local, curved coordinates defined by the same shape functions as used in the function approximation (isoparametric system) by Irons [Ref. 48] and the subsequent use of *numerical integration* permit just such a development; see Fig. 2. Today, such elements are used extensively in two- and three-dimensional analysis [Refs. 47, 49, 50]. Figs. 3(a) and

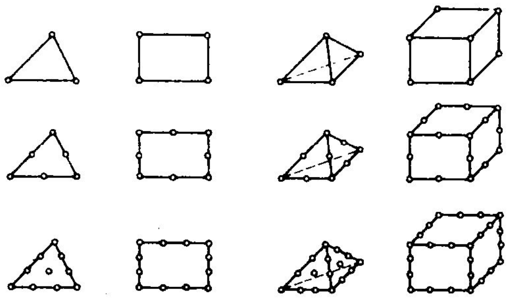

FIG. 1. SOME TWO- AND THREE-DIMENSIONAL ELEMENTS

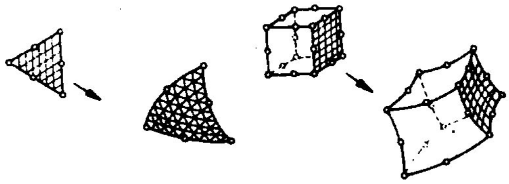

FIG. 2. DISTORTION OF SIMPLE FORM BY USE OF CURVILINEAR COORDINATES

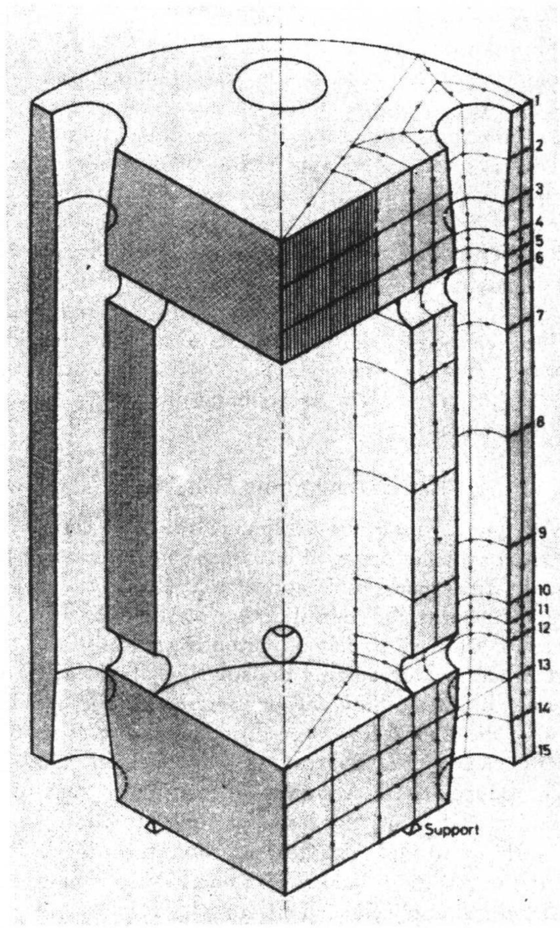

FIG. 3a. CURVED ISOPARAMETRIC ELEMENTS IN A THREE-DIMENSIONAL STRESS ANALYSIS OF A PRESSURE VESSEL

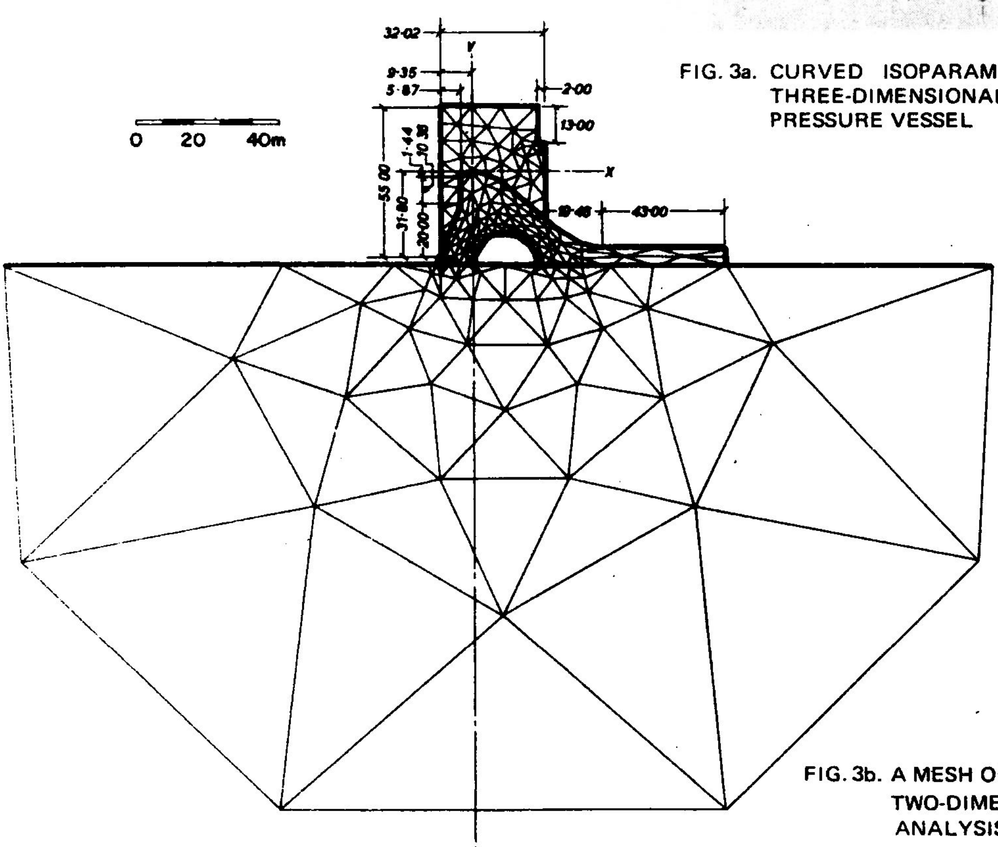

FIG. 3b. A MESH OF SIMPLE TRIANGLES IN A TWO-DIMENSIONAL STRESS ANALYSIS OF A DAM

(b) show typical subdivision of "practical" domain by simple and curved elements (the first with four, the second with 60 degrees of freedom).

The use of numerical integration introduced first in such elements is now widespread. Not only can element properties be evaluated rapidly by its use but many opportunities for algebraic or computer errors are eliminated.

In some problems, the definition of the functional requires additionally the continuity of first or even higher derivatives of the function between elements. Such situations are typical of plate and shell problems in elasticity or viscous flow problems in fluid mechanics. Here, the generation of appropriate shape functions becomes more difficult. Successful avoidance of this continuity already has been mentioned [Refs. 26, 31, 32] in the context of plate bending, and same elements have been used in viscous flow solutions [Ref. 51]. However, the achievement of compatibility can be accomplished in two ways: first, by the use of singular functions shown in context of simple plate elements by Clough, et al., [Refs. 52, 53], Bazeley, et al., [Ref. 26], and Veubeke [Ref. 19], or second, by introduction of higher order continuities (second derivatives) at nodes first done by Bogner and Schmit [Ref. 54]. Indeed, the latter procedure is gaining popularity in various complex plate elements currently advocated [Refs. 55-59].

Similar problems are encountered in shell analysis, and here the reader can trace the development from simplest approximation by flat or conical surfaces [Refs. 60, 61] to recent, sophisticated curved shell elements [Refs. 52, 62, 63]. A full bibliography can be found in the text [Ref. 25].

Once again, the isoparametric concept can be used and curved shell elements obtained by degeneration of full three-dimensional continua as shown by Ahmad, et al., [Refs. 64, 65]. Indeed, it is interesting to remark on the side how here the more fundamentally correct formulation avoids the pitfalls introduced by the physically dubious approximations which, perforce, have to be introduced into classical plate and shell theory.

Nonlinearity and Dynamic Problems

Problems of solid mechanics with nonlinearity due to material or geometric causes or problems with dynamic terms obviously can be dealt with in a similar manner using a finite element discretization as when other methods (such as finite differences) accomplish the same purposes. In the first, some form of iterative approach has to be adopted, while in the latter, eigenvalue or stepwise time integration methods have to be used. With the power of solving complex, linear, boundary-value problems available, it is obvious that such applications are now becoming so numerous that a comprehensive review is not practicable. Once again the simple and direct approach used in the physical formulation of finite elements proves invaluable in treatment of such comprexities. Indeed, the close visualization of the phenomena allows the development of iterative processes which mathematically might not be obvious.

With nonlinear material properties, for instance, systematic adjustments of initial strains, initial stresses or of elastic constants in the linear elastic problem allow rapid convergence of the essentially nonlinear equations to be achieved. That mathematically such approaches often are identical to Newton-Raphson or other known techniques does not diminish their usefulness in choosing the most rapid convergence path. The [Refs. 25, 45, 66-77, 104] are but a few selected examples of the wide impact of finite element processes on tackling such situations in which the physical properties range from idealized metal plasticity, through various creep phenomena to some very specialized properties exhibited by rock and concrete. A review summarizing the various approaches used is given in [Ref. 77].

Essentially similar approaches can be followed in geometric nonlinearity problems such as arise in stability and large deflection considerations of slender structures. Although the classical, linearized stability eigenvalue problem – so popular in analytical literature – was the first target [Refs. 78-82], now the possibilities of following through the large deformation influences are at last open [Refs. 40, 83-89].*

Indeed, the feasibility of combining nonlinear material properties with large deformation behavior are indicated by Marcal [Ref. 86], and many further studies in this context doubtless will appear. Some interesting applications of the process to large strain problems have been given by Oden [Refs. 90, 91].

The literature on dynamic application and in particular on natural frequency studies of problems by the finite element method is so large that it will not be cited here. Indeed, the omission is justifiable inasmuch as standard methods are usually used once the discretization has been accomplished. For some typical application, the text by Zienkiewicz [Ref. 25] or the Proceedings of Conferences on Matrix Methods in Structural Mechanics held at Wright-Patterson AFB, Ohio, in 1965 and 1968 can be consulted. Much, however, remains to be done in the general study of transient response of linear and nonlinear systems, and doubtless more will be heard in that context in the near future.

The Future Path

From this brief survey it should be apparent that (a) the method of a wide general applicability and that (b) most of the problems to which it has been applied are of the structural-solid mechanics type.

Indeed, perhaps without undue exaggeration, it may be said that with present day size of computers solutions can be obtained to all solid mechanics/structural situations on a practical basis. While further work doubtless will continue here, a path of diminishing returns has

*Thus the mathematically interesting but physically untenable bifurcation stability studies may finally be replaced by criteria much closer to reality.

been reached at least in matters of basic formulation. A spate of papers on new element forms is unlikely to result in dramatic improvements (although here perhaps the problem of shell structures should be excluded). Perhaps the most justifiable work is now in the study and introduction of realistic material parameters and constitutive description, and as this work progresses many novel applications will be made.

The position is not yet similar in other fields of engineering activity. Some simple application of the possibilities in the context of fluid mechanics, [Refs. 92-94], heat transfer and electromagnetic phenomena have been mentioned [Refs. 12, 13, 15, 16, 37, 51, 95, 96]. Some more complex problems involving structure-fluid interaction, etc., have been formulated and solved [Refs. 97-100]. Others, like the flow phenomena with vorticity, solution of Navier-Stokes equation and flow of non-Newtonian substance, are under study at the present time at the author's institution (Swansea) and elsewhere. It is in such new applications and others yet not formulated that the potential expansion lies in the general applied mechanics sense.

All this having been said, from the engineers' point of view, matters of maximum efficiency are vital, even for problems which now can be solved. Here, ideally, use of the powerful method should result in daily design improvement. Questions of *economy* and *ease of access* are paramount. Already much has been done at various university centers where such work is going on and at large industrial organizations. (References [101] and [102], for instance, demonstrate the advances in equation solving procedures so essential to the whole process.) Automatic mesh generation process, easy data handling methods, efficient programs capable of breaking the "cost barriers" and finally simple output devices are items on which efforts must be continued. If further breakthrough are to be expected, it is in this area that they will occur.

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