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BISPECTRAL METHODS OF SIGNAL PROCESSING

APPLICATIONS IN RADAR, TELECOMMUNICATIONS AND DIGITAL IMAGE RESTORATION

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Authors

Prof. Dr. Alexander V. Totsky
National Aerospace University
Dept. of Receiving, Transmitting & Signal Processing
Chkalova Str. 17
Kharkov
61070
Ukraine
e-mail: totskiy@xai.edu.ua

Prof. Dr. Alexander A. Zelensky
National Aerospace University
Dept. of Receiving, Transmitting & Signal Processing
Chkalova Str. 17
Kharkov
61070
Ukraine
e-mail: azelens@mail.ru

Prof. Dr. Victor F. Kravchenko
Russian Academy of Sciences
Kotel'nikov Institute of Radio Engineering and Electronics of RAS
Mokhovaya 11–7
Moscow
125009
Russia
e-mail: kvf-ok@mail.ru

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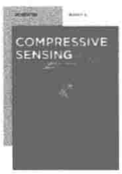

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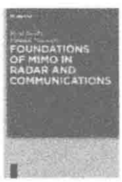

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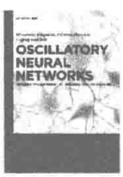

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Introduction

Try extracting harmony contained in chaos

Bispectrum-based techniques of signal processing and higher-order statistical analysis [1–4] have attracted the attention of many researchers in the 1980s of the previous century as prospective addition and sometimes as an alternative tool to common second-order spectral-correlation analysis widely spread for different applications like radar, sonar, pattern recognition, digital communications, nondestructive control, biomedical engineering, and so on. A. W. Lohmann and B. Wirnitzer (see, e.g., [1]) were one of the first researchers who addressed optical and astronomical applications of triple correlation and bispectrum. Their pioneer investigations performed in astronomy stimulated other applications like sonars, biomedical engineering, nondestructive control, radars, communications, image processing, and so on, that have been generalized in [2] by C. L. Nikias and M. R. Raghuveer. Theoretical aspects of bispectrum-based signal processing were summarized by J. M. Mendel in [3] and by C. L. Nikias; A. P. Petropulu in their book [4]. Recently, the amount of publications dedicated to higher-order statistics and bispectral analysis has increased radically.

For almost twenty years the researchers were basically dealing with the theoretical aspects of triple correlation and bispectrum estimation. This peculiarity can be explained by the necessity of extensive computations for processing and storage of multidimensional data required for higher-order statistical analysis. Lately, with increased computing power, the interest in practical applications of bispectrum-based signal processing has increased. This is explained with known inherent advantages of bispectrum that radically differ it from common second-order power spectrum estimation. These advantages are as follows: possibility to extract phase-coupled contributions contained in processed signals, immunity to zero mean noise with symmetric probability density function, random signal shift and jitter invariance properties, as well as preservation of phase information contained in processed data.

However, many both theoretical and practical questions yet remain unclear. They are:

- The statistical properties of noisy bispectrum and triple correlation estimates have not been analyzed in detail; effective ways for their improvement have not been thoroughly studied yet.
- The phase wrapping problem still exists in signal processing and it influences the quality of bispectrum-based signal waveform restoration.
- Recently, a lot of attention has been paid to the problem of bispecrum-based 1-D signal processing; however, it is natural to expect promising results in case of bispectrum-based 2-D noisy and jittery image processing and digital image reconstruction.

- The problem of detection of deterministic signal embedded in noise by using novel higher-order test statistics formed at the matched filter is of particular interest for digital communications and radar applications.
- Both sea and vegetation clutter suppression in radar applications by using bispectral-based signal processing can provide improving target recognition and classification performance.
- Bispectrum-based approach is able to extract the phase coupling contribution contained in nonstationary and multicomponent radar signals backscattered by moving targets observed in vegetation clutter. This will make it possible to obtain novel information features for better radar target recognition and classification.
- The performance of bispectrum-based signal processing is not thoroughly investigated for such real-life situations like small input signal-to-noise ratios (SNR) and a small number of observed realizations.

These problems are addressed and discussed below in this book.

The goal of the book is the theoretical and experimental study of bispectrum-based techniques and algorithms developed for digital processing of signals and images. The basic application is radars of different types intended for detection and automatic recognition of aerial, ground moving and naval targets, surveillance in vegetation and sea clutter. Other applications like digital wireless communications and digital image processing are considered as well.

The book contains four Chapters.

Chapter 1 gives theoretical background and deals with analysis of basic properties of bispectrum and triple correlation function and accuracy of bispectrum estimation. Some particular aspects like phase unwrapping are discussed. Statistical study of bispectral estimates contaminated by interferences is performed and extreme accuracy is analytically defined by Cramér–Rao criterion.

Chapter 2 is devoted to combined bispectrum-filtering techniques that exploit positive features of bispectrum and filtering, linear and nonlinear, nonadaptive and adaptive. Non-Gaussianity and nonstationarity of fluctuations observed in bispectral domain induced by leakage of input noise is demonstrated. This serves the purpose of designing novel adaptive filters suitable for this application.

Reconstruction of images distorted by jitter and additive noise is considered in Chapter 3. Removal of jitter observed with influence of mixture of additive Gaussian and impulsive noise is studied. It is shown that additive predistortions provide both removal of phase ambiguity and jitter by using bispectrum-based image reconstruction. Approach based on multiplicative predistortions allows decreasing distortions in restored images as compared with additive predistortions. Optimal additive and multiplicative predistortion function parameters are evaluated and analyzed.

A novel bispectral technique for signal detection and discrimination is suggested in Chapter 4 by using test detection statistics computed in the form of peak values of the third-order moment functions. A novel encoding concept using frequency diversity strategy and bispectrum-based signal processing are suggested for wireless communication systems. According to the proposed approach, binary data are transmitted by using a pair of mutually orthogonal triplet-signals contained phase-coupled frequency tones. Novel third-order test detection statistics evaluated in the form of triplet-signal bimagnitude peaks are suggested for detection and discrimination of received triplet-signals in noisy and fading communication radio links. Radar applications of bispectrum are considered in Chapter 4. It contains experimental results for coastal naval, ground surveillance and aerial target recognition and classification radars.

Contributions of the authors are represented as follows. A. V. Totsky is contributed to all Chapters of the book. A. A. Zelensky is contributed to the sub-Chapters 1.1, 1.2, 2.1, 2.2, 2.4, 3.1 and 4.2. V. F. Kravchenko is contributed to the sub-Chapter 1.4.

Contents

1 General properties of bispectrum-based digital signal processing

1.1 General properties of cumulant and moment functions

Common power or energy spectral density estimation is a well-known and widely spread tool for random signal analysis. Ensemble averaged Fourier magnitude spectrum density does not contain any information about behavior of a centered random process in the frequency domain since the spectral components are statistically independent in different observed realizations. In this case, the energy distribution of statistically independent spectral components must be estimated since the energy content does not depend on the phase relationships for separate frequencies. Indeed, for the processes containing independent spectral components, the energy spectrum estimate is the exhaustive characteristic conventionally used in spectral analysis of such processes.

In several practical applications of signal processing, an analyzed process can contain phase coupled spectral contributions. Study of these spectral correlation relationships can give us very useful and important information for correct understanding, analysis and description of physical effects that cause a given process. Note that such information about phase coupling is irretrievably lost in common energy spectrum estimates.

Cumulant function and cumulant spectrum estimation can serve as a very useful and promising tool for signal analysis and processing. Cumulant-based approach has several important and attractive benefits as compared with energy spectrum estimation. These benefits are listed and described below.

First, consider mathematical description of cumulant spectra for a real-valued stationary and discrete-valued process given by the time series as $\{x(i), i = 0, 1, 2, \ldots\}$. The joint cumulants $c_x^{(r)}(\tau_1, \tau_2, \ldots, \tau_{r-1})$ of r^{th} order can be defined as

$$c_x^{(r)} = c_x(\tau_1, \tau_2, \ldots, \tau_{r-1}) = -j^r \left[\frac{\partial^r \ln \Theta(\omega_1, \omega_2, \ldots, \omega_r)}{\partial\omega_1 \partial\omega_2 \ldots \partial\omega_r} \right]_{\omega_1=\omega_2=\ldots=\omega_r=0}, \tag{1.1.1}$$

where $\Theta(\omega_1, \omega_2, \ldots, \omega_r) = \langle \exp[j(\omega_1 x_1 + \omega_2 x_2 + \ldots + \omega_r x_r)] \rangle_x$ is the multidimensional characteristic function; $\omega_1, \omega_2, \ldots, \omega_r$ are the angular frequencies; $j = \sqrt{-1}$; $\langle \ldots \rangle_x$ denotes ensemble averaging procedure; and $\tau_1, \tau_2, \ldots, \tau_{r-1}$ are the time shifts.

The cumulants (1.1.1) serve as the characteristic of the probability distribution and they can be represented by the following coefficients in Taylor series for the function $\ln \Theta(\omega)$ in the neighborhood of the point of origin

$$\ln \Theta(\omega) = \sum_{r=1}^{\infty} \frac{c_x^{(r)}}{r!} (j\omega)^r . \tag{1.1.2}$$

The joint moments $m_x(\tau_1, \tau_2, \ldots, \tau_{r-1})$, defined for a stationary process $\{x(i), i = 0, 1, 2, \ldots\}$ differ from the cumulants (1.1.1) as follows

$$m_x^{(r)} = m_x(\tau_1, \tau_2, \ldots, \tau_{r-1}) = \langle x(i)x(i+\tau_1)x(i+\tau_2)\ldots x(i+\tau_{r-1})\rangle_x = -j^k \left[\frac{\partial^r \Theta(\omega_1, \omega_2, \ldots, \omega_r)}{\partial\omega_1 \partial\omega_2 \ldots \partial\omega_r}\right]_{\omega_1=\omega_2=\ldots=\omega_r=0} . \tag{1.1.3}$$

The joint moments (1.1.3) can be defined by the expansion coefficients of the characteristic function $\Theta(\omega)$ in Taylor series in the neighborhood of the point of origin as

$$\Theta(\omega) = 1 + \sum_{r=1}^{\infty} \frac{m_x^{(r)}}{r!}(j\omega)^r . \tag{1.1.4}$$

The relationships between the joint cumulants (1.1.1) and the joint moments (1.1.3) in the origin under assumption that $\tau_1 = \tau_2 = \ldots \tau_{r-1} = 0$ can be written by the following formulas

$$\begin{aligned}
c_x^{(1)} &= m_x^{(1)} = \langle x(i)\rangle , \\
c_x^{(2)} &= m_x^{(2)} - \left(m_x^{(1)}\right)^2 , \\
c_x^{(3)} &= m_x^{(3)} - 3m_x^{(1)}m_x^{(2)} + 2\left(m_x^{(1)}\right)^3 , \\
c_x^{(4)} &= m_x^{(4)} - 3\left(m_x^{(2)}\right)^2 - 4m_x^{(1)}m_x^{(3)} + 12\left(m_x^{(1)}\right)^2 m_x^{(2)} - 6\left(m_x^{(1)}\right)^4 .
\end{aligned} \tag{1.1.5}$$

For the case of a zero-mean process, that is, for $m_x^{(1)} = \langle x(i)\rangle = 0$, the formulas (1.1.5) transform to the following structure

$$\begin{aligned}
c_x^{(1)} &= m_x^{(1)} = 0 , \\
c_x^{(2)} &= m_x^{(2)} = \langle x(i)\rangle^2 = \sigma^2 , \\
c_x^{(3)} &= m_x^{(3)} = \langle x^3(i)\rangle , \\
c_x^{(4)} &= m_x^{(4)} - 3(m_x^{(2)})^2 = \langle x^4(i)\rangle - 3(\sigma^2)^2 ,
\end{aligned} \tag{1.1.6}$$

where σ^2 is the variance of a process under consideration.

Let us consider a real-valued discrete and zero-mean process $\{x(i), i = 0, 1, 2, \ldots, I-1\}$, $\langle x(i)\rangle = 0$. The relationships between the moment and cumulant functions for this zero-mean process can be described by the following formulas

$$\langle x(i)x(i+k)\rangle = m_x^{(2)}(k) = c_x^{(2)}(k) , \tag{1.1.7a}$$

$$\langle x(i)x(i+k)x(i+l)\rangle = m_x^{(3)}(k,l) = c_x^{(3)}(k,l) , \tag{1.1.7b}$$

$$\begin{aligned}
\langle x(i)x(i+k)x(i+l)x(i+m)\rangle &= m_x^{(4)}(k,l,m) = c_x^{(4)}(k,l,m) + c_x^{(2)}(k)c_x^{(2)}(m-l) \\
&\quad + c_x^{(2)}(k)c_x^{(2)}(m-k) + c_x^{(2)}(m)c_x^{(2)}(l-k) ,
\end{aligned} \tag{1.1.7c}$$

where k, l and m are the shift indices.

The formula (1.1.7a) describes the relationship between the second-order statistics and it defines conventional autocorrelation function. Note that second-order moment and cumulant functions are equal to each other in this case.

The formula (1.1.7b) describes the relationship between the third-order statistics and it defines triple or third-order autocorrelation function. It should be noted that the third-order moments and cumulants are equal to each other in this case.

According to the formula (1.1.7c) defining the relationship existing between the fourth-order statistics, the fourth-order moment function is not equal to the fourth-order cumulant function.

Spectral density of the r^{th} order called as polispectrum or cumulant spectrum $C_x(\omega_1, \omega_2, \ldots, \omega_{r-1})$ of a process $\{x(i), i = 0, 1, 2, \ldots, I - 1\}$ can be defined by the following multidimensional Fourier transform of the r^{th} order cumulant $c_x^{(r)}(\tau_1, \tau_2, \ldots, \tau_{r-1})$ as

$$C_x^{(r)}(\omega_1, \omega_2, \ldots, \omega_{r-1},) = \sum_{\tau_1=-\infty}^{+\infty} \cdots \sum_{\tau_{r-1}=-\infty}^{+\infty} c_x^{(r)}(\tau_1, \tau_2, \ldots, \tau_{r-1}) \exp[-j(\omega_1\tau_1 + \omega_2\tau_2 + \ldots + \omega_{r-1}\tau_{r-1})]. \tag{1.1.8}$$

The generalized formula (1.1.8) allows to define the energy spectrum $P_x(\omega)$ (for $r = 2$), the bispectrum $B_x(\omega_1, \omega_2)$ (for $r = 3$) and the trispectrum $T_x(\omega_1, \omega_2, \omega_3)$ (for $r = 4$), respectively, in the forms:

$$P_x(\omega) = \sum_{l=-\infty}^{+\infty} c_x^{(2)}(l) \exp[-j(\omega l)] , \tag{1.1.9a}$$

$$B_x(\omega_1, \omega_2) = \sum_{l_1=-\infty}^{+\infty} \sum_{l_2=-\infty}^{+\infty} c_x^{(3)}(l_1, l_2) \exp[-j(\omega_1 l_1 + \omega_2 l_2)] , \tag{1.1.9b}$$

$$T_x(\omega_1, \omega_2, \omega_3) = \sum_{l_1=-\infty}^{+\infty} \sum_{l_2=-\infty}^{+\infty} \sum_{l_3=-\infty}^{+\infty} c_x^{(4)}(l_1, l_2, l_3) \exp[-j(\omega_1 l_1 + \omega_2 l_2 + \omega_3 l_3)]. \tag{1.1.9c}$$

The expressions (1.1.9a–b) contain the cumulant functions whose properties are interesting and worth considering in detail.

(1) If x_i, $i = 1, 2, \ldots, K$ is a sequence of random variables and $\alpha_i = 1, 2, \ldots, K$ are some constant values, then

$$c(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_K x_K) = \left(\prod_{i=1}^{K} \alpha_i\right) c(x_1, x_2, \ldots, x_K) . \tag{1.1.10}$$

(2) Permutation property for random variables

$$c(x_1, x_2, \ldots x_K) = c(x_{i1}, x_{i2}, \ldots, x_{iK}) , \tag{1.1.11}$$

where $(i_1, i_2, \ldots, i_K)$ is the permutation index $(1, 2, \ldots, K)$.

(3) Additivity property of the cumulants for their arguments

$$c(x+y, z_1, z_2, \ldots, z_K) = c(x, z_1, z_2, \ldots, z_K) + c(y, z_1, z_2, \ldots, z_K)\,, \quad (1.1.12)$$

which signifies that the cumulant of the sum of arguments is equal to the sum of the cumulants of the separate arguments.

(4) If α is a constant value then

$$c(\alpha + x_1, x_2, \ldots, x_K) = c(x_1, x_2, \ldots, x_K)\,. \quad (1.1.13)$$

(5) In the case when the random variables x_i, $i = 1, 2, \ldots, K$ and y_i, $i = 1, 2, \ldots, K$ are independent, we have

$$c(x_1 + y_1, x_2 + y_2, \ldots, x_K + y_K) = c(x_1, x_2, \ldots, x_K) + c(y_1, y_2, \ldots, y_K)\,. \quad (1.1.14)$$

Assume that an observed process is $z(i) = x(i) + n(i)$, $i = 1, 2, \ldots, K$, and $x(i)$ and $n(i)$ are the independent processes. According to the property (1.1.14), one can obtain

$$c_z^{(K)}(l_1, l_2, \ldots, l_{K-1}) = c_x^{(K)}(l_1, l_2, \ldots, l_{K-1}) + c_n^{(K)}(l_1, l_2, \ldots, l_{K-1})\,. \quad (1.1.15)$$

If one of the process, for example, $n(i)$ is Gaussian, then under condition of $K \geq 3$, $c_n^{(K)}(l_1, l_2, \ldots, l_{K-1}) = 0$ we obtain

$$c_z^{(K)}(l_1, l_2, \ldots, l_{K-1}) = c_x^{(K)}(l_1, l_2, \ldots, l_{K-1})\,. \quad (1.1.16)$$

The latter expression (1.1.16) demonstrates important insensitivity property to the additive Gaussian noise valid for the cumulants the order of which is equal or more than three. From the practical point of view of signal processing in additive Gaussian noise environment, the cumulant estimates permit to separate non-Gaussian signal from additive Gaussian noise and, hence, to increase signal-to-noise (SNR) ratio.

Below, we will pay attention to the third-order statistics from the point of view of their application in digital signal and image processing. For this purpose, first, we will consider the general properties of triple correlation and bispectrum and the techniques used for their estimation.

1.2 Triple correlation function and bispectrum

One of the main motivations referred to using bispectrum-based signal processing is the following. Bispectrum density estimate or third-order cumulant spectrum estimate, opposite to the energy spectrum estimate, not only allows to describe the statistical properties of an observed process more correctly and completely, but also to extract novel information features such as spectral component correlation relationships. Moreover, bispectrum estimate allows extracting the phase relationships existing between the spectral components contained in the process under study. Therefore, the main difference of bispectrum from energy spectrum is in preservation of phase information contained in a process and the possibility to recover this important information. Already, this promising peculiarity of bispectrum has contributed to wide usage of the bispectrum analysis and bispectral estimation techniques for digital signal processing. Permanent growth in the interest in bispectrum analysis is accompanied by appearance of a great number of publications.

Consider the benefits of bispectrum analysis more in detail. One of the most promising bispectrum property usually used for recovering a signal embedded in Gaussian noise, is the tendency to zero the bispectrum of an interference having a symmetrical probability density function (PDF) . This property provides robustness of the bispectrum-based filtering techniques regarding additive Gaussian interference in radar [5–8], astronomy [9–11], underwater acoustics [12–14], and biomedical [15, 16] signal processing systems, as well as in digital image processing systems [17–20].

Bispectral analysis can serve as quite a sensitive and precise tool permitting to define and measure the deviation of the observed process from Gaussian distribution, that is, to estimate non-Gaussianity. This property seems to be very useful in noisy-like processes in machine diagnostics systems [21], underwater acoustic systems [12], nondestructive monitoring [22], and biomedical diagnostics [16].

Let us consider the properties of bispectrum for a real-valued stationary discrete process $\{x^{(m)}(i)\}$ given by finite number of samples $i = 0, 1, 2, \ldots, I-1$ and observed with a finite sequence of $m = 1, 2, \ldots, M$ realizations.

Common autocorrelation discrete function $R_x(k)$ belonging to the class of second-order statistics can be written as a function of a single variable

$$R_x(k) = \left\langle \sum_{i=0}^{I-1} \left[x^{(m)}(i) - E\right] \left[x^{(m)}(i+k) - E\right] \right\rangle_\infty , \tag{1.2.1}$$

where $k = -I+1, \ldots, I-1$ is the temporal or spatial shift index; $\langle\ldots\rangle_\infty$ denotes ensemble averaging assuming that number of accumulated realizations tends to infinity, that is, $M \to \infty$; $E = \langle(1/I)\sum_{i=0}^{I-1} x^{(m)}(i)\rangle_\infty$ is the mean value; $R_x(0) = \sigma_x^2 = \langle\sum_{i=0}^{I-1} [x^{(m)}(i) - E]^2\rangle_\infty$ is the variance.

Autocorrelation function $R_x(k)$ (1.2.1) has the following symmetry property

$$R_x(k) = R_x(-k) . \tag{1.2.2}$$

According to the Wiener–Khinchin theorem, the spectral density $P_x(p)$ can be defined in the form of the following discrete direct Fourier transform (DFT)

$$P_x(p) = \sum_{k=-\infty}^{k=+\infty} R_x(k) \exp(-j2\pi kp) , \tag{1.2.3}$$

or by

$$P_x(p) = \left\langle \dot{X}^{(m)}(p)\dot{X}^{*(m)}(p) \right\rangle_\infty , \tag{1.2.4}$$

where $p = -I + 1, \ldots, I + 1$ is the frequency sample index; $\dot{X}^{(m)}(p) = \sum_{i=0}^{I-1} x^{(m)}(i)$ $\exp(-j2\pi ip)$ is the complex-valued DFT computed for m^{th} arbitrary realization; $*$ denotes complex conjugation.

It should be taken into account one more time that Fourier phase spectrum information is irretrievably lost in the spectral density (1.2.4).

Own autocorrelation function corresponds to each concrete signal, but not inversely. It is impossible to restore signal shape by autocorrelation function as it is impossible to restore the shape of some plane figure by its known square.

Triple autocorrelation function (TAF) $R_x(k, l)$ represents the third-order statistic. TAF is a function of two variables and it can be represented in the discrete form as

$$R_x(k,l) = \left\langle \sum_{i=0}^{I-1} \left[x^{(m)}(i) - E\right] \left[x^{(m)}(i+k) - E\right] \left[x^{(m)}(i+l) - E\right] \right\rangle_\infty , \tag{1.2.5}$$

where $k = -I + 1, \ldots, I - 1$ and $l = -I + 1, \ldots, I - 1$ are the independent shift indices.

Note that the TAF (1.2.5) possesses the following symmetry property [2]

$$R_x(k,l) = R_x(l,k) = R_x(l-k,-k) = R_x(k-l,-l) = R_x(-k,l-k) . \tag{1.2.6}$$

According to the definitions given in [1, 2], bispectrum is the 2-D DFT of TAF. Unlike the real-valued spectral density (1.2.3) and (1.2.4), bispectrum (or bispectral density) is the complex-valued function $\dot{B}_x(p, q)$ of two independent frequencies p and q that can be written as the following 2-D discrete DFT of TAF (1.2.5)

$$\dot{B}_x(p,q) = \sum_{k=-I+1}^{I-1} \sum_{l=-I+1}^{I-1} R_x(k,l) \exp[-j2\pi(kp + lq)] , \tag{1.2.7a}$$

or as

$$\dot{B}_x(p,q) = \left\langle \dot{X}^{(m)}(p)\dot{X}^{(m)}(q)\dot{X}^{*(m)}(p+q) \right\rangle_\infty = \left\langle \dot{X}^{(m)}(p)\dot{X}^{(m)}(q)\dot{X}^{(m)}(-p-q) \right\rangle_\infty , \tag{1.2.7b}$$

where $\dot{B}_x(p,q) = |\dot{B}_x(p,q)| \exp[j\gamma_x(p,q)]$; $|\dot{B}_x(p,q)|$ and $\gamma_x(p,q)$ are the magnitude bispectrum (bimagnitude) and phase bispectrum (biphase), respectively; $p = -I + 1, \ldots, I - 1$ and $q = -I + 1, \ldots, I - 1$ are the frequency indices.

Comparing the spectral (1.2.4) and bispectral (1.2.7b) densities allows noting that spectral density $P_x(p)$ is the ensemble averaging performed for the multiplication of two complex conjugated functions corresponding to the same frequency p and bispectral density $\dot{B}_x(p,q)$ is the ensemble averaging of triple product of three complex-valued functions related to three different frequencies: p, q and $p + q$.