PHY

国外物理名著系列 20

（影印版）

Applications of Random Matrices in Physics

随机矩阵在物理学中的应用

É.Brézin V.Kazakov D.Serban
P.Wiegmann A.Zabrodin

科学出版社
www.sciencep.com

国外物理名著系列 28

Applications of Random Matrices in Physics

随机矩阵在物理学中的应用

E.Brézin V.Kazakov D.Serban
P.Wiegmann A.Zabrodin

国外物理名著系列　20

Applications of Random Matrices in Physics

随机矩阵在物理学中的应用

É. Brézin V. Kazakov D. Serban P. Wiegmann A. Zabrodin

科学出版社
北京

图字:01-2007-3523

É. Brézin, V. Kazakov, D. Serban, P. Wiegmann, A. Zabrodin: Applications of Random Matrices in Physics

图书在版编目(CIP)数据

随机矩阵在物理学中的应用=Applications of Random Matrices in Physics:英文/(德)布拉钦(Brézin,É.)等编著.—影印本.—北京:科学出版社,2008
(国外物理名著系列;20)
ISBN 978-7-03-022626-6

Ⅰ.随… Ⅱ.布… Ⅲ.随机矩阵-应用-物理学-英文 Ⅳ.04

中国版本图书馆CIP数据核字(2008)第122385号

责任编辑:王飞龙 胡 凯 鄢德平/责任印制:赵德静/封面设计:陈 敬

科学出版社 出版
北京东黄城根北街16号
邮政编码:100717
http://www.sciencep.com

双青印刷厂 印刷

科学出版社发行 各地新华书店经销

*

2008年8月第 一 版 开本:B5(720×1000)
2008年8月第一次印刷 印张:33
印数:1~2 500 字数:631 000

定价:98.00元

(如有印装质量问题,我社负责调换〈双青〉)

国外物理名著系列序言

对于国内的物理学工作者和青年学生来讲，研读国外优秀的物理学著作是系统掌握物理学知识的一个重要手段。但是，在国内并不能及时、方便地买到国外的图书，且国外图书不菲的价格往往令国内的读者却步，因此，把国外的优秀物理原著引进到国内，让国内的读者能够方便地以较低的价格购买是一项意义深远的工作，将有助于国内物理学工作者和青年学生掌握国际物理学的前沿知识，进而推动我国物理学科研和教学的发展。

为了满足国内读者对国外优秀物理学著作的需求，科学出版社启动了引进国外优秀著作的工作，出版社的这一举措得到了国内物理学界的积极响应和支持，很快成立了专家委员会，开展了选题的推荐和筛选工作，在出版社初选的书单基础上确定了第一批引进的项目，这些图书几乎涉及了近代物理学的所有领域，既有阐述学科基本理论的经典名著，也有反映某一学科专题前沿的专著。在选择图书时，专家委员会遵循了以下原则：基础理论方面的图书强调“经典”，选择了那些经得起时间检验、对物理学的发展产生重要影响、现在还不“过时”的著作（如：狄拉克的《量子力学原理》）。反映物理学某一领域进展的著作强调“前沿”和“热点”，根据国内物理学研究发展的实际情况，选择了能够体现相关学科最新进展，对有关方向的科研人员和研究生有重要参考价值的图书。这些图书都是最新版的，多数图书都是 2000 年以后出版的，还有相当一部分是 2006 年出版的新书。因此，这套丛书具有权威性、前瞻性和应用性强的特点。由于国外出版社的要求，科学出版社对部分图书进行了少量的翻译和注释（主要是目录标题和练习题），但这并不会影响图书“原汁原味”的感觉，可能还会方便国内读者的阅读和理解。

“他山之石，可以攻玉”，希望这套丛书的出版能够为国内物理学工作者和青年学生的工作和学习提供参考，也希望国内更多专家参与到这一工作中来，推荐更多的好书。

杨国桢

中国科学院院士

中国物理学会理事长

Preface

Random matrices are widely and successfully used in physics for almost 60-70 years, beginning with the works of Wigner and Dyson. Initially proposed to describe statistics of excited levels in complex nuclei, the Random Matrix Theory has grown far beyond nuclear physics, and also far beyond just level statistics. It is constantly developing into new areas of physics and mathematics, and now constitutes a part of the general culture and curriculum of a theoretical physicist.

Mathematical methods inspired by random matrix theory have become powerful and sophisticated, and enjoy rapidly growing list of applications in seemingly disconnected disciplines of physics and mathematics.

A few recent, randomly ordered, examples of emergence of the Random Matrix Theory are:

- universal correlations in the mesoscopic systems,
- disordered and quantum chaotic systems;
- asymptotic combinatorics;
- statistical mechanics on random planar graphs;
- problems of non-equilibrium dynamics and hydrodynamics, growth models;
- dynamical phase transition in glasses;
- low energy limits of QCD;
- advances in two dimensional quantum gravity and non-critical string theory, are in great part due to applications of the Random Matrix Theory;
- superstring theory and non-abelian supersymmetric gauge theories;
- zeros and value distributions of Riemann zeta-function, applications in modular forms and elliptic curves;
- quantum and classical integrable systems and soliton theory.

In these fields the Random Matrix Theory sheds a new light on classical problems.

On the surface, these subjects seem to have little in common. In depth the subjects are related by an intrinsic logic and unifying methods of theoretical physics. One important unifying ground, and also a mathematical basis for the Random Matrix Theory, is the concept of integrability. This is despite the fact that the theory was invented to describe randomness.

The main goal of the school was to accentuate fascinating links between different problems of physics and mathematics, where the methods of the Random Matrix Theory have been successfully used.

We hope that the current volume serves this goal. Comprehensive lectures and lecture notes of seminars presented by the leading researchers bring a reader to frontiers of a broad range of subjects, applications, and methods of the Random Matrix Universe.

We are gratefully indebted to Eldad Bettelheim for his help in preparing the volume.

EDITORS

Contents

RANDOM MATRICES AND NUMBER THEORY

J. P. Keating
School of Mathematics,
University of Bristol,
Bristol, BS8 1TW
UK

1. Introduction

My purpose in these lecture notes is to review and explain some recent results concerning connections between random matrix theory and number theory. Specifically, I will focus on how random matrix theory has been used to shed new light on some classical problems relating to the value distributions of the Riemann zeta-function and other L-functions, and on applications to modular forms and elliptic curves.

This may all seem rather far from Physics, but, as I hope to make clear, the questions I shall be reviewing are rather natural from the random-matrix point of view, and attempts to answer them have stimulated significant developments within that subject. Moreover, analogies between properties of the Riemann zeta function, random matrix theory, and the semiclassical theory of quantum chaotic systems have been the subject of considerable interest over the past 20 years. Indeed, the Riemann zeta function might be viewed as one of the best testing grounds for those theories.

In this introductory chapter I shall attempt to paint the number-theoretical background needed to follow these notes, give some history, and set some context from the point of view of Physics. The calculations described in the later chapters are, as far as possible, self-contained.

1.1 Number-theoretical background

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

for Re$s > 1$, where p labels the primes, and then by analytic continuation to the rest of the complex plane. It has a single simple pole at $s = 1$, zeros at $s = -2, -4, -6$, etc., and infinitely many zeros, called the *non-trivial zeros*,

E. Brezin et al. (eds.), Applications of Random Matrices in Physics, 1–32.

in the *critical strip* $0 < \text{Re}s < 1$. It satisfies the *functional equation*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2}$$

The *Riemann Hypothesis* states that all of the non-trivial zeros lie on the *critical line* $\text{Re}s = 1/2$ (i.e. on the symmetry line of the functional equation); that is, $\zeta(1/2 + it) = 0$ has non-trivial solutions only when $t = t_n \in \mathbb{R}$ [33]. This is known to be true for at least 40% of the non-trivial zeros [6], for the first 100 billion of them [36], and for batches lying much higher [29].

In these notes I will, for ease of presentation, assume the Riemann Hypothesis to be true. This is not strictly necessary – it simply makes some of the formulae more transparent.

The mean density of the non-trivial zeros increases logarithmically with height t up the critical line. Specifically, the *unfolded* zeros

$$w_n = t_n \frac{1}{2\pi} \log \frac{|t_n|}{2\pi} \tag{3}$$

satisfy

$$\lim_{W\to\infty} \frac{1}{W} \# \{w_n \in [0, W]\} = 1; \tag{4}$$

that is, the mean of $w_{n+1} - w_n$ is 1.

The zeta function is central to the theory of the distribution of the prime numbers. This fact follows directly from the representation of the zeta function as a product over the primes, known as the *Euler product*. Essentially the nontrivial zeros and the primes may be thought of as Fourier-conjugate sets of numbers. For example, the number of primes less than X can be expressed as a harmonic sum over the zeros, and the number, $N(T)$, of non-trivial zeros with heights $0 < t_n \leq T$ can be expressed as a harmonic sum over the primes. Such connections are examples of what are generally called *explicit formulae*. Ignoring niceties associated with convergence, the second takes the form

$$N(T) = \overline{N}(T) - \frac{1}{\pi} \sum_p \sum_{r=1}^{\infty} \frac{1}{rp^{r/2}} \sin(rT \log p), \tag{5}$$

where

$$\overline{N}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O\left(\frac{1}{T}\right) \tag{6}$$

as $T \to \infty$. This follows from integrating the logarithmic derivative of $\zeta(s)$ around a rectangle, positioned symmetrically with respect to the critical line and passing through the points $s = 1/2$ and $s = 1/2 + iT$, using the functional equation. (Formulae like this can be made to converge by integrating both sides against a smooth function with sufficiently fast decay as $|T| \to \infty$.)

It will be a crucial point for us that the Riemann zeta-function is but one example of a much wider class of functions known as L-functions. These L-functions all have an Euler product representation; they all satisfy a functional equation like the one satisfied by the Riemann zeta-function; and in each case their non-trivial zeros are subject to a generalized Riemann hypothesis (i.e. they are all conjectured to lie on the symmetry axis of the corresponding functional equation).

To give an example, let

$$\chi_d(p) = \left(\frac{d}{p}\right) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod{p} \text{ solvable} \\ 0 & \text{if } p|d \\ -1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod{p} \text{ not solvable} \end{cases} \tag{7}$$

denote the Legendre symbol. Then define

$$\begin{aligned} L_D(s, \chi_d) &= \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}, \end{aligned} \tag{8}$$

where the product is over the prime numbers. These functions form a family of L-functions parameterized by the integer index d. The Riemann zeta-function is itself a member of this family.

There are many other ways to construct families of L-functions. It will be particularly important to us that elliptic curves also provide a route to doing this. I will give an explicit example in the last chapter of these notes.

1.2 History

The connection between random matrix theory and number theory was first made in 1973 in the work of Montgomery [28], who conjectured that

$$\lim_{W \to \infty} \frac{1}{W} \#\{w_n, w_m \in [0, W] : \alpha \leq w_n - w_m < \beta\} = \int_\alpha^\beta \left(\delta(x) + 1 - \frac{\sin^2(\pi x)}{\pi^2 x^2}\right) dx. \tag{9}$$

This conjecture was motivated by a theorem Montgomery proved in the same paper that may be restated as follows:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n,m \leq N} f(w_n - w_m) = \int_{-\infty}^{\infty} f(x) \left(\delta(x) + 1 - \frac{\sin^2(\pi x)}{\pi^2 x^2}\right) dx \tag{10}$$

for all test functions $f(x)$ whose Fourier transforms

$$\widehat{f}(\tau) = \int_{-\infty}^{\infty} f(x)\exp(2\pi i x\tau)dx \tag{11}$$

have support in the range (-1, 1) and are such that the sum and integral in (10) converge. The generalized form of the Montgomery conjecture is that (10) holds for all test functions such that the sum and integral converge, without any restriction on the support of $\widehat{f}(\tau)$. The form of the conjecture (9) then corresponds to the particular case in which $f(x)$ is taken to be the indicator function on the interval $[\alpha, \beta)$ (and so does not fall within the class of test functions covered by the theorem).

The link with random matrix theory follows from the observation that the pair correlation of the nontrivial zeros conjectured by Montgomery coincides precisely with that which holds for the eigenvalues of random matrices taken from either the Circular Unitary Ensemble (CUE) or the Gaussian Unitary Ensemble (GUE) of random matrices [27] (i.e. random unitary or hermitian matrices) in the limit of large matrix size. For example, let A be an $N \times N$ unitary matrix, so that $A(A^T)^* = AA^\dagger = I$. The eigenvalues of A lie on the unit circle; that is, they may be expressed in the form $e^{i\theta_n}$, $\theta_n \in \mathbb{R}$. Scaling the eigenphases θ_n so that they have unit mean spacing,

$$\phi_n = \theta_n \frac{N}{2\pi}, \tag{12}$$

the two-point correlation function for a given matrix A may be defined as

$$R_2(A;x) = \frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{k=-\infty}^{\infty} \delta(x + kN - \phi_n + \phi_m), \tag{13}$$

so that

$$\frac{1}{N}\sum_{n,m} f(\phi_n - \phi_m) = \int_0^N R_2(A;x)f(x)dx. \tag{14}$$

$R_2(A;x)$ is clearly periodic in x, so can be expressed as a Fourier series:

$$R_2(A;x) = \frac{1}{N^2}\sum_{k=-\infty}^{\infty} |\mathrm{Tr}A^k|^2 e^{2\pi i kx/N}. \tag{15}$$

The CUE corresponds taking matrices from $U(N)$ with a probability measure given by the normalized Haar measure on the group (i.e. the unique measure that is invariant under all unitary transformations). It follows from (15) that the CUE average of $R_2(A;x)$ may be evaluated by computing the corresponding average of the Fourier coefficients $|\mathrm{Tr}A^k|^2$. This was done by Dyson

[14]:

$$\int_{U(N)} |\mathrm{Tr}A^k|^2 d\mu_{Haar}(A) = \begin{cases} N^2 & k=0 \\ |k| & |k| \le N \\ N & |k| > N. \end{cases} \tag{16}$$

There are several methods for proving this. One reasonably elementary proof involves using Heine's identity

$$\int_{U(N)} f_c(\theta_1, \ldots, \theta_N) d\mu_{Haar}(A) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} f_c(\theta_1, \ldots, \theta_N) \det(e^{i\theta_n(n-m)}) d\theta_1 \cdots d\theta_N \tag{17}$$

for *class functions* $f_c(A) = f_c(\theta_1, \theta_2, \ldots, \theta_N)$ (i.e. functions f_c that are symmetric in all of their variables) to give

$$\int_{U(N)} |\mathrm{Tr}A^k|^2 d\mu_{Haar}(A) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_j \sum_l e^{ik(\theta_j - \theta_l)} \times \begin{vmatrix} 1 & e^{-i\theta_1} & \cdots & e^{-i(N-1)\theta_1} \\ e^{i\theta_2} & 1 & \cdots & e^{-i(N-2)\theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i(N-1)\theta_N} & e^{i(N-2)\theta_N} & \cdots & 1 \end{vmatrix} d\theta_1 \cdots d\theta_N. \tag{18}$$

The net contribution from the diagonal ($j = l$) terms in the double sum is N, because the measure is normalized and there are N diagonal terms. Using the fact that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}, \tag{19}$$

if $k \ge N$ then the integral of the off-diagonal terms is zero, because, for example, when the determinant is expanded out and multiplied by the prefactor there is no possibility of θ_1 cancelling in the exponent. If $k = N - s$, $s = 1, \ldots, N-1$, then the off-diagonal terms contribute $-s$; for example, when $s = 1$ only one non-zero term survives when the determinant is expanded out, multiplied by the prefactor, and integrated term-by-term – this is the term coming from multiplying the bottom-left entry by the top-right entry and all of the diagonal entries on the other rows. Thus the combined diagonal and off-diagonal terms add up to give the expression in (16), bearing in mind that when $k = 0$ the total is just N^2, the number of terms in the sum over j and l.

Heine's identity itself may be proved using the Weyl Integration Formula [35]

$$\int_{U(N)} f_c(A) d\mu_{Haar}(A) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f_c(\theta_1, \ldots, \theta_N) \times \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_N \quad (20)$$

for class functions $f_c(A)$, the Vandermonde identity

$$\prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 = \det \left[M M^\dagger \right] \quad (21)$$

where

$$M = \begin{pmatrix} 1 & 1 & \cdots \\ e^{i\theta_1} & e^{i\theta_2} & \cdots \\ \vdots & \ddots & \cdots \\ e^{i(N-1)\theta_1} & e^{i(N-1)\theta_2} & \cdots \end{pmatrix}, \quad (22)$$

the fact that

$$\det \left[M M^\dagger \right] = \det \left[\sum_{\ell=1}^{N} e^{i\theta_\ell (n-m)} \right], \quad (23)$$

and then by performing elementary manipulations of the rows in this determinant.

The Weyl Integration formula will play a central role in these notes. One way to understand it is to observe that, by definition, $d\mu_{Haar}(A)$ is invariant under $A \to \tilde{U} A \tilde{U}^\dagger$ where $\tilde{U}$ is any $N \times N$ unitary matrix, and that A can always be diagonalized by a unitary transformation; that is, it can be written as

$$A = U \begin{pmatrix} e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\theta_N} \end{pmatrix} U^\dagger, \quad (24)$$

where U is an $N \times N$ unitary matrix. Therefore the integral over A can be written as an integral over the matrix elements of U and the eigenphases θ_n. Because the measure is invariant under unitary transformations, the integral over the matrix elements of U can be evaluated straightforwardly, leaving the integral over the eigenphases (20).

Henceforth, to simplify the notation, I shall drop the subscript on the measure $d\mu(A)$ – in all integrals over compact groups the measure may be taken to be the Haar measure on the group.