Industrial Application of Electromagnetic Computer Codes

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PREFACE

During the last decade a new generation of software tools has evolved in computational electromagnetics. Both analytical methods and particularly numerical techniques have improved considerably, leading to an extended range of capabilities and an increased applicability of both dedicated and general purpose computer codes.

It is the intention of this volume to review the state of the art in electromagnetic analysis and design, and to describe the fundamentals and the advances in theoretical/numerical approaches coupled with practical solutions for static and time-dependent fields.

In this context, the book illustrates the effectiveness of numerical techniques and associated computer codes in solving real electromagnetic field problems. In addition, it demonstrates the usefulness of modern codes for the analysis of many industrial practical cases. In particular, solutions of magnetostatic and magnetodynamic problems applied to electrical machines, induction heating, non-destructive testing, fusion reactor technology and other industrial are presented and discussed.

The present volume reflects and combines the lectures which are organized in the frame of the Eurocourse programme at JRC Ispra under the sponsorship of the Institute for Systems Engineering and Informatics (ISEI). It is hoped that in this context the Institute and particularly the Systems Engineering & Reliability (SER) Division can play a stimulating role in sponsoring and promoting the diffusion of knowledge in novel areas of computer and information science.

Giuseppe Volta
SER Division Head

Robert W. Witty
ISEI Director

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OVERVIEW OF THE 'STATE OF THE ART' IN ELECTROMAGNETIC ANALYSIS AND DESIGN

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ABSTRACT. The use of computer software for the design of electromagnetic devices is now almost universal [1,2,3,4,5,6,7]. This lecture presents an overview of the major advances in the subject and attempts to identify the principal limitations. Particular attention is given to the computing environment, 3-D algorithms for electromagnetic field computation and in the use of parallel architectures.

1. Introduction

The widespread use of numerical techniques today demonstrates the great importance of the subject. Efficient solutions can be obtained for a wide range of problems well beyond the scope of analytical methods. The use of numerical methods has overcome the limitations imposed by analytic methods, i.e. their restriction to homogenous, linear and steady-state problems. For example, it is now routine to compute the magnetic forces acting on the components of an electrical machine, taking into account the three dimensional geometry including the slots, conductors, etc. as well as the saturation effects of the material. It is now common practice for engineers to solve many of the complex problems arising during the design phase of sophisticated modern electromagnetic devices by using largely automatic and general purpose procedures; examples include electrical devices of all types as well as the large magnets used in medical diagnostic equipment, particle accelerators, electron beam lenses, fusion magnets and industrial processes—the range of applications is extensive. This success is due largely to the enormous advances that have been made in the power of the digital computer; and although the basis of numerical analysis extends back a long way in history, there is also a steady stream of new developments allowing for more and more difficult problems to be solved. If this activity continues at the present rate it is expected that in the next few years the efficiency of computer methods will reach the stage when genuine computer aided design (CAD) procedures are practical for three dimensional systems, i.e. design *lay-out* software will be integrated with electromagnetic field analysis programs[8].

Numerical analysis is the technique for solving mathematical problems by numerical approximation. By this means solutions are arrived at with the aid of rational numbers and, inevitably, since digital computers are used to perform the computations, only a finite set of

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these numbers will be available. In addition to this *rounding error* the size of the computer memory (i.e. the number of *words*) will limit the complexity of the problem that can be solved which in turn means that it will be impossible to achieve a perfect representation of the geometry; thus the actual problem will usually be replaced by a *computer model*, built by the engineer to represent the critical features under investigation. The use of a numerical technique will introduce further *modelling errors* called discretisation or truncation errors arising when the mathematical description, a continuous partial differential equation, is replaced by an approximate numerical description characterised by discrete points.

The discretisation and solution of continuum problems, in this case electrical engineering devices, have been approached differently by mathematicians and engineers, see the text by Zienkiewicz([9] 1977, page 1). The former have concentrated on general techniques, directly applicable to the field equations, such as finite differences approximations and weighted residual procedures, whereas the latter have often used a more intuitive approach exploiting analogues between real discrete elements and finite regions of the continuum. Indeed, electrical engineers have often used circuit analogues to model their problem both experimentally and numerically; and before the widespread use of the digital computer many other analogues were used experimentally such as *resistive paper* and *electrolytic tanks*. The intuitive, direct anology, approach by engineers, particularly in the area of structural mechanics lead to the development of the Finite Element Method [10]; and by 1960 this technique was widely used in other disciplines. It is no accident that these advances came at this time since in the early 1960's occured the rapid development of the digital computer as a universal tool for engineers. In the meantime electrical engineers had, in the main, followed and applied the developments in Finite Differences, now a highly developed discipline of mathematics, and were able to write elegant computer codes particularly for simple static two dimensional configurations with linear media—for example, Hornsby(1967) [11] at CERN developed a successful code used extensively in the design of particle physics magnets.

An important milestone in the solution of electromagnetics field problems came in 1963 with the seminal work of Winslow [12] at the Lawrence Livermore Laboratory California; he developed a discretisation scheme based on an irregular grid of plane triangles, not only by using a generalised finite difference scheme but also by introducing a variational principle which he showed led to the same result. This latter approach can be seen as being equivalent to the Finite Element Method and is accordingly one of the earliest examples of this technique used for electromagnetics. The resulting computer code 'TRIM' [13] and its later descendent 'POISSON' [14] have been used all over the world. Finite difference techniques continued to be applied by electrical engineers throughout the 1960's and early 1970's, notably the work of Trutt at Delaware [15], Erdelyi *et al* at Colorado [16], and Viviani [17] *et al* in Genoa, and in three dimensions by Muller and Wolf at AEG Telefunken, Germany [18].

However by the early 1970's the Finite Element Method was under scrutiny by the mathematicians and substantial generalisations were made [19] and many cross links were established with earlier work on variational calculus and generalised weighted residuals [20]. The important advantages of finite elements over finite differences were being exploited, i.e. the ease of modelling complicated boundaries and the extendibility to higher order approximations and in 1970 came the first application of the method to rotational electrical machines by Chari and Silvester [21]. From this time on the use of the method became widespread leading to generalised applications for time dependent and three dimensional problems by the group at Rutherford Laboratory with the production of the codes PE2D and TOSCA [22,23].

A parallel development to the above has been with integral methods; these integral formulations, unlike differential formulations which solve the defining partial differential equations (e.g. Poisson's Equation), use the corresponding integral equation forms, e.g. equations based on Gauss' theorem. The moment method is an example of an integral formulation, see the text by Harrington (1968) [24] for a basic treatment; yet another class of an integral procedures are the so called boundary element methods [25,26] based on applications of Greens integral theorems. Whilst these methods are often difficult to apply they can produce accurate economic solutions and have been used extensively in certain static and high frequency problems, an example of a general purpose program, first developed in 1971, is the magnetisation integral equation code GFUN which was specifically designed for three dimensional static problems [27].

In this lecture the author gives a personal view on the present status of electromagnetic computing. The following topics will be considered:

- Evolution of Code Development
- Limitations in Contemporary Codes Unsolved Problems
- A Modern Computing Environment CAD system for the designer
- Economics of 3-D Computation
- 3-D Field Formulations
- Impact of Parallel Processing
- The Way Ahead

2. Evolution of Code Development

It is instructive to see over the years how the frontiers of tractable problems have been extended and to observe what ground still needs to be covered. In Table 1 are listed the dates when codes with the indicated functionality became generally available. It will be noticed that there was a significant use of integral equation techniques in the early days leading to both two and three dimensional codes. This development was curtailed mainly due to the excessive computing power needed to solve systems of equations whose matrix of coefficients is fully populated. This subject will be returned to later when parallel computers are discussed. The finite element method used to solve the differential forms of the field equations has been strikingly sucessful leading to general purpose software. Thus by the year 1990 3-D solutions can be achieved by electromagnetic device designers in industry for a fairly large range of problems.

Time Evolution of Code Development							
Function	1960	1965	1970	1975	1980	1985	1990
Statics	D2Df I2D	D2D I3D	D2Dnp I3Dn	D2Da	D3Dna	D2Dse	D2Dv
Steady State A.C.			D2D I2D	D2Dn*	D3D†	D2De	D3D
Transient			I2D	D2D	D2Dn	D3D†	D3D
Full Maxwell						D2D	D3D
Motion					D2D‡	D3D‡	D3D
Coupled Problems				D2Df	D2D		D3D

Key

* Approximate model; ‡Uni-directional velocity; †Restricted formulation

D2D Differential 2-D
D3D Differential 3-D
I2D Integral 2-D
I3D Integral 3-D

n...non-linear
p...permanent magnet
s...sacalar hysteresis
e...error analysis

a...anisotropy
f...finite difference
v...vector hysteresis

Table 1: *The Evolution of Fuctionality in EM Codes*

3. Limitations in Contemporary Codes

Although the functionality of contemporary codes has grown over the years, nevertheless as the functionality increases so do the requirements of users. The following list outlines areas where considerable work is still required:–

- Vector Hysteresis
  Major criticisms of numerical methods have been made because, far too often, inadequate material models are used. Considerable research is currently underway to develop better material models, including hysteresis effects [28,29,30]. Nevertheless the sheer complexity involved in keeping track of minor loops remains a daunting task.

- Far Field
  A major consideration in any electromagnetic field analysis is the placement of the far-field boundary. In many cases the natural boundary of a magnet is essentially at infinity although in practice the presence of remote objects and their possible affect on the field shape will have to be taken into account. Special care will be needed with all methods based on solving the differential form of the defining equations. A number of approaches to this problem have been described in the literature [31]:

  1. Terminating the field at a sufficiently large distance.
     This of course begs the question and at the very least will require several solutions in order to achieve confidence. A useful technique here is to solve two problems at each trial with Dirichlet and Neumann boundary conditions respectively in order to bound the solution.
  2. To use special finite element basis functions which have the correct assymptotic behaviour for large distances[32]. The major advantage of using special finite elements is that the matrix size and bandwidth are not significantly increased, but on the other hand some knowlege of how the field decays is needed in order to fix scaling parameters.This approach has been sucessfully implemented in a number of cases [31].
  3. A number of methods based on the idea of a super global element to model the exterior region. The method of recursive ballooning [33] in which the global element is generated iteratively by succesively adding concentric rings of *scaled elements* to embrace the finite element model. At each step the adjacent nodes between rings are removed so there are only nodes on the original and far boundaries. Boundary conditions at the far boundary can now be applied. The method is not so effective in 3-D.
     A major difficulty in these global element methods is that the matrix structure is strongly affected and that the band-width will increase to a point where the matrix becomes essentially full and an advantage of the differential approach disappears.
  4. Mapping techniques have also been extensively used to transform the exterior infinite space to a finite space. The classical Kelvin transformation (see Kellogg [34], page 232) is an example of this in which the transformation $rr' = a^2$

where r and r' are the inverse points with respect to a sphere of radius a. An inversion in a sphere is one-to-one except that the centre of the sphere of inversion has no corresponding point. The neighbourhood of the origin maps into a set of points at a large distance—into an infinite domain. This transformation has been used in a number of finite element systems to model the infinite domain in which the exterior space to a sphere surrounding the actual model is solved as an interior problem by means of the Kelvin transformation. The nodes of the two spaces, now bounded by spheres, are connected by forcing their solution values to be identical [35], [36], [37].

5. The need for special methods is obviated totally if integral methods are used which is one of the major advantages of this approach and can be recommended for small to medium sized problem or larger if computers with parallel architectures are available.

- **3D Field Formulations**
  This fundamental topic was discussed in the previous section and remains the most important single issue in computational electromagnetics.

- **Moving Boundaries**
  The analysis of moving sytems with a high magnetic *Reynold's Number* can lead to singularly perturbed problems. Thus the standard Poisson type equations are modified to the form, $\epsilon\nabla^2\phi + v\mu\sigma\frac{\partial\phi}{\partial x} = f$, where $0 < \epsilon \ll 1$. This can cause instability in the solution if standard finite element methods are used (see reference [38], page 40). Considerable success with problems of this type has been achieved in the area of computational fluid mechanics which has carried over to electromagnetics [39,40]. At this point it may be pertinent to ask whether integral methods may offer significant advantages here since free-space between moving conducors would not require meshing!

- **Coupled Problems**
  Real industrial problems involve other technologies in addition to electromagnetics—e.g. thermal, structural and fluid. There are many inportant questions to be discussed in this context. One such is to what extent can solving the separate systems in series be used, i.e. using sequential algorithms which minimise the size of the solution space but may, in practice, fail to converge. Or, alternatively accept the enormous costs of parallel, fully coupled analysis. There is a growing body of significant work on coupled phenomena reported in the literature, see reference [7], pages 536-582.

- **Optimisation**
  It is, of course, the engineer's proper role to provide the creativity and not to waste too much time on *what if?* experiments or, at least, he should consider if an algorithm can be devised in order to provide answers rather than *knob turning* which misses or could never reach a solution in time—the number of states for a binary choice escalates as 2^n where n is a number of variables.

  Put another way, the process of design often requires the determination of boundary values that produce a desired field, i.e. it is the solution to the inverse problem that is needed. As an example consider the problem of determining a boundary shape to produce a uniform field in the gap of an electromagnet. In the region under the pole, the magnetic field will, in particular, depend upon the geometry, material and current sources. In particular, if shims are to be used, the optimisation problem is to determine the values of their width and depth that produce a constant value of B_p inside a specified domain. One way of doing this is to solve the field equations by finite elements or boundary elements. To do this suppose only the two limiting values of each shim parameter are explored, i.e. for two shims there will be 8 states, thus there will be $2^8 = 256$ cases to run! This clearly is unfair because trends will be observed on the way and experience will eliminate many of the trials, never the less a lot of computation and time is to be expected. In cases like this automatic optimisation should be considered; there are several methods reported in the literature including least square techiques [27,41], evolution strategies [42,43] and simulated annealing [44]. The near hysterical *hype* of artificial intelligence is also having an impact on electromagnetics design but at present this remains shallow and suspect. Nevertheless, the related and practical procedures using knowledge base methods and *expert systems* has already influenced new research projects in integrated design analysis [8].

4. A Modern Computing Environment

4.1. THE DESIGN PROCESS

Not only have there been advances in the functionality of numerical codes but also the computing environment that these codes work in has changed beyond recognition from the days of the mainframe and batch processing. In Figure 1 is shown the normal procedure of using a field code for design. The designer iterates toward his design solution by a more or less heuristic method(*cut and try?*) supplemented by the engineers creativity and experience. In order to carry out this process efficiently the software system has three usual stages of pre-processing, solution processing and post-processing.

4.2. A CAD SYSTEM FOR DESIGN

In a modern environment the user should expect the following facilities:

- Fast Solutions
- Error Estimation
- Practical Ergonomics for Data Input
- Meaningful Post-Processing
- Automatic Discretisation

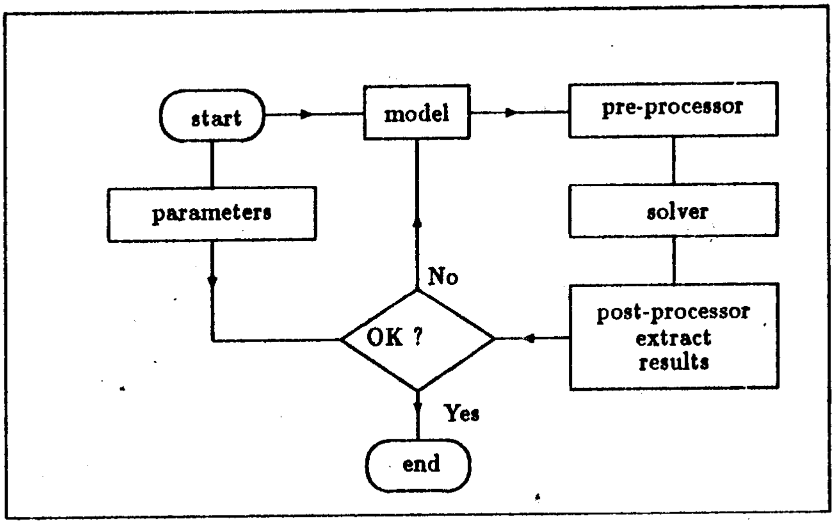


Figure 1: *Flow Diagram for Heuristic Design.*

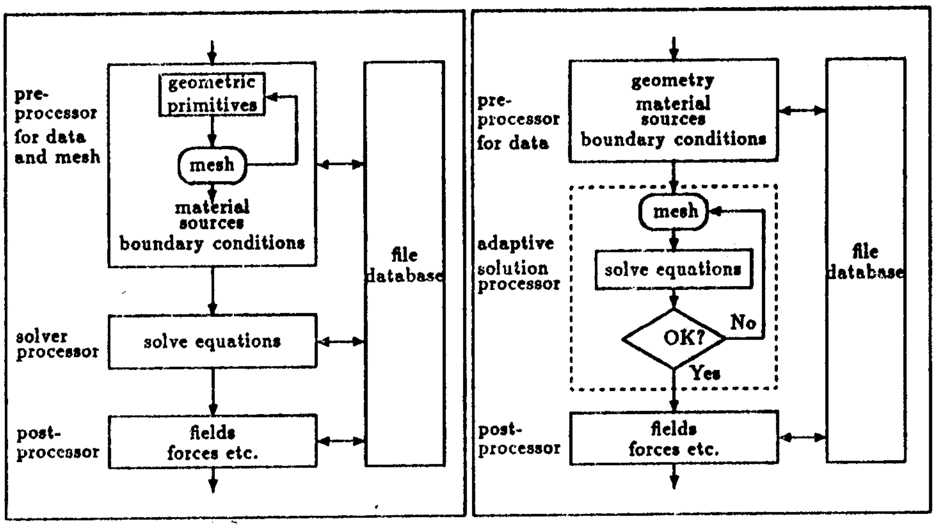
Thus the following components are needed:

- Data Base
- Pre-Processor
- Adaptive Solver
- Post-Processor
- Knowledge Base

In Figure 2 is shown two versions of a design environment. Figure 2(a) shows the normal system in which the pre-processor includes data input, model building and mesh generation. User controlled meshing is extremely tedious. Although fully automated meshing is now a practical possibility it needs to be combined with error estimation in order to allow the generation of optimal meshes. This approach is now becoming quite common for 2-D systems [45,46] and can be expected in 3-D systems before long. Figure 2(b) shows a more ideal system in which the solution processor includes an adaptive mesh generator controlled by *a posteriori* error estimation [45].

5. Economics of 3D Field Computation

Despite advances in solution techniques and computer hardware 3D computation is prohibitive for many applications.This is mainly due to additional complexity involved in moving from 2-D to 3-D. There are two main aspects:



(a) Normal System (b) Ideal System

Figure 2: *Components of CAD System*

1. The 3D geometry itself leads to a larger sytem matrix and and to a far more complex mesh generation task.
2. Vector nature of the field for time dependent problems causes a ≥ 3 fold increase in the system matrix size.

This has a dramatic effect on the solver compute time T, i.e.

$$T = \alpha n^a + \beta n^b + \gamma n^c f(n)$$

where the three terms refer to the source field, matrix set-up, and matrix solution times respectively. The machine dependent constants usually satisfy:

$$\alpha \gg \beta \gg \gamma$$

Consider two examples:

1. Differential Operator
   using a preconditioned conjugate gradient method,e.g. ICCG
   $a = 1, b = 1, c = 0, f \equiv nlog(n)$
   i.e. for a problem with 10×10 nodes in xy plane:

2D Scalar	3D Vector	nlog(n) ratio
10x10	3x10x10x10	~ 50

2. Integral Operator using Gaussian Elliminination, then $a = 1, b = 2, c = 3, f \equiv 1$

2D Scalar	3D Vector	n^3 ratio
10x10	3x10x10x10	~ 27000

The above examples clearly demomstrate why the differential method has been the most sucessful to date!. However to be fair *like is not being compared with like* since integral equation solutions need far fewer unknowns. Furthermore if parallel computers are used a significant speed-up is to be expected since the integral formulation is intrinsically parallel (see section 8.).

6. 3-D Field Formulations

6.1. QUASI-STATIC ELECTROMAGNETIC FIELD EQUATIONS

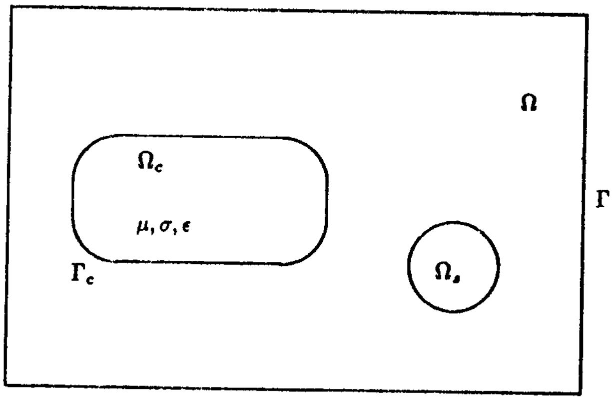


Figure 3: *Simple Model Configuration for Eddy Currents*

The basic equations describing electromagnetic fields are listed here, without detailed explanations, to remind the reader of the concepts and nomenclature. For the purpose of introducing the field equations it is convenient to consider the elementary model problem shown in Figure (3) in which a volume of conducting material Ω_c, with magnetic permeability μ and electrical conductivity σ, bounded by a surface Γ_c, is contained within a global volume of free space Ω bounded by a surface Γ which, furthermore, may be extended to infinity if required. The global region may also contain a number of prescribed conductor sources Ω_s which do not intersect Ω_c. This configuration arises in a very large number of applications of practical importance in industry and problems of this type have been the starting point for many developers of computer algorithms. However care is needed if Ω contains multiply connected regions.
It is known that if the dimensions of the regions Ω_c and Ω_s are small compared with the wavelength of the prescribed fields then the displacement current term in Maxwell's equations will be small compared to the free current density **J** and there will be, essentially, no radiation (see Stratton, page 277) [47]. This regime means a return to the pre-Maxwell

field equations, the so called quasi-static case, where Ampere's law is a good approximation. In this situation the field equations can be approximated by:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \quad (Gauss's\ Law) && (1)\\ \nabla \cdot \mathbf{B} &= 0 && (2)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad (Faraday's\ Law) && (3)\\ \nabla \times \mathbf{H} &= \mathbf{J} \quad (Ampere's\ Law\ ) && (4) \end{aligned}$$

where **D**,**B**,**E**,**H** are the usual field vectors, ρ and **J** the free charge and current densities respectively [47]. The field vectors are not independent since they are further related by the material constitutive properties;

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} && (5)\\ \mathbf{B} &= \mu \mathbf{H} && (6) \end{aligned}$$

where ϵ and μ are the material permittivity and permeability respectively. The current density in a conductor moving with relative velocity **v** is generated by the Lorentz force and is given by:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (Ohm's\ Law) \tag{7}$$

where σ is the material conductivity. In practice μ and σ may often be field dependent quantities, and furthermore, some materials will exhibit both anisotropic and hysteretic effects. The current continuity condition follows from Eq. (4), and is:

$$\nabla \cdot \mathbf{J} = 0. \tag{8}$$

The four field vectors must satisfy the following conditions at the interfaces between regions of different material properties;

$$\begin{aligned} (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} &= 0 && (9)\\ (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} &= \omega && (10)\\ (\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{n} &= \mathbf{K} && (11)\\ (\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n} &= 0 && (12) \end{aligned}$$

where **K** and ω are the surface current and charge densities respectively. These relations follow directly from the limiting forms of field equations, Equations (1) to (4), applied at the interfaces.Furthermore, for the quasi-static case, Eq. (8) implies continuity of the normal component of current density at material interfaces.

6.2. MAGNETIC VECTOR POTENTIAL A

Since the field vector **B** satisfies a zero divergence condition, it can be expressed in terms of a vector potential **A** as follows:

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{13}$$

and then, from the field equations,Eq. (3), it follows that,

$$\nabla \times (\mathbf{E} - \frac{\partial}{\partial t}\mathbf{A}) = 0, \tag{14}$$

and hence by integrating to give,

$$\mathbf{E} = -(\frac{\partial}{\partial t}\mathbf{A} + \nabla V), \tag{15}$$

where V is a scalar potential. Neither **A** nor V are completely defined since the gradient of an arbitrary scalar function can be added to **A** and the time derivative of the same function can be subtracted from V without affecting the physical quantities **E** and **B**. These changes to **A** and V are the so called gauge transformations, and uniqueness is usually ensured by specifying the divergence (gauge) of **A** and sufficient boundary conditions. Thus in region Ω_c the field equations in terms of **A** and V are as follows:

$$\nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} + \sigma(\frac{\partial \mathbf{A}}{\partial t} + \nabla V) = \mathbf{J}_s \tag{16}$$

$$\nabla \cdot \sigma(\frac{\partial \mathbf{A}}{\partial t} + \nabla V) = 0 \tag{17}$$

and, in the global region where $\sigma = 0$ and $\nabla \times \mathbf{H} = \mathbf{J}_s$, reduces to

$$\nabla \cdot \mu \nabla \phi = 0, \tag{18}$$

where ϕ is the reduced magnetic scalar potential with $\mathbf{H} = \mathbf{H}_s - \nabla\phi$ for a source field $\mathbf{H}_s$.

At points just inside a conductor the continuity conditions imply

$$\mathbf{J_n} = -\sigma(\frac{\partial \mathbf{A}}{\partial t} + \nabla \mathbf{V}) \cdot \mathbf{n} = 0 \tag{19}$$

$$\frac{\partial \mathbf{A_n}}{\partial t} + \frac{\partial V}{\partial n} = 0 \tag{20}$$

at conductor surfaces, and at interfaces, across which the conductivity changes from σ_1 to σ_2 implies that,

$$\sigma_1(\frac{\partial}{\partial t}\mathbf{A}_1 + \nabla V_1) \cdot \mathbf{n} = \sigma_2(\frac{\partial}{\partial t}\mathbf{A}_2 + \nabla V_2) \cdot \mathbf{n}. \tag{21}$$

Are Equations (16) and (17) sufficient? It is clear that Eq. (17) is a consequence of taking the divergence of Eq. (16) and is not, therefore, independent. Some investigators [48,49] have obtained unique solutions to Equations (16) and (17), as they stand, but they show that the uniqueness depends upon the particular numerical procedures used. This leaves a flexibility of the system unused thus is it possible to use this flexibility to advantage? In any case it is necessary to specify the divergence (gauge) of **A** and appropriate boundary conditions to