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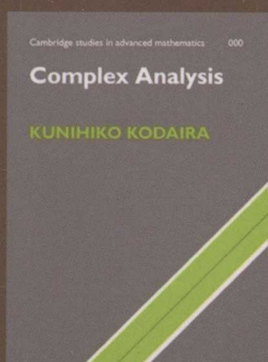
Complex Analysis

小平邦彦复分析

（英文版）

[日] 小平邦彦 著



人民邮电出版社
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北　京

图书在版编目（CIP）数据

小平邦彦复分析＝Complex Analysis：英文/（日）小平邦彦著. —北京：人民邮电出版社，2008.6
（图灵原版数学 · 统计学系列）
ISBN 978-7-115-17840-4

I.小… II.小… III.复分析－教材－英文 IV.O174.5

中国版本图书馆CIP数据核字（2008）第035540号

内容提要

本书讲述了复变函数的经典理论．作者用易于理解的方式严密介绍基础理论，强调几何观点，避免了一些拓扑学难点．书中首先从拓扑上较简单的情形论证了柯西积分公式，并引出连续可微函数的基本性质．然后阐述共形映射、解析延拓、黎曼映射定理、黎曼面及其结构，以及闭黎曼面上的解析函数等．书中包含大量的图示和丰富的例子，并附有习题，可以帮助读者增强对课程的理解．

本书可作为高等院校理工科专业复分析的入门教材，也可作为更高级学习研究的参考书．

图灵原版数学 . 统计学系列

小平邦彦复分析（英文版）

◆ 著　［日］小平邦彦
责任编辑　张继发　明永玲

◆ 人民邮电出版社出版发行　北京市崇文区夕照寺街14号
邮编　100061　电子函件　315@ptpress.com.cn
网址　http://www.ptpress.com.cn
北京隆昌伟业印刷有限公司印刷
新华书店总店北京发行所经销

◆ 开本：700×1000　1/16
印张：25.75
字数：440千字　2008年6月第1版
印数：1－3 000册　2008年6月北京第1次印刷

著作权合同登记号　图字：01-2008-0975号

ISBN 978-7-115-17840-4/O1

定价：59.00元

读者服务热线：（010）88593802　印装质量热线：（010）67129223

反盗版热线：（010）67171154

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Preface

This book aims to give a clear explanation of classical theory of analytic functions; that is, the theory of holomorphic functions of one complex variable. In modern treatments of function theory it is customary to call a function holomorphic if its derivative exists. However, we return to the old definition, calling a function holomorphic if its derivative exists *and* is continuous, since we believe this is a more natural approach.

The first difficulty one encounters in writing an introduction to function theory is the topology involved in Cauchy's Theorem and Cauchy's integral formula. In the first chapter of the book we prove the latter in a topologically simple case, and from that result we deduce the basic properties of holomorphic functions. In the second chapter we prove the general version of Cauchy's Theorem and integral formula. I have tried to replace the necessary topological considerations with elementary geometric considerations. This way turned out to be longer than I expected, so that in the original Japanese three-volume edition I had to end Volume 2 before Chapter 5 was completed. My original intention was to present classical many-valued analytic functions, in particular the Riemann surface of an algebraic function, and to introduce the general concept of a Riemann surface as its generalization. Now, with the appearance of the complete Japanese edition in a single volume, the link between the theory of Riemann surfaces and function theory is restored.

Similarly, for the theory of Riemann surfaces, with the assumption that the topology of curved surfaces is known, the plan was to introduce the content of Weyl's book: *The Concept of Riemann Surfaces. Part II: Functions on Riemann Surfaces*, but that would have been counter to the original policy of replacing the topological approach with elementary geometrical considerations. Thus, in Chapter 7, I have tried to illuminate topological characteristics of compact Riemann surfaces by using Riemann's mapping theorem. Consequently, Chapter 7 became longer than was planned, so Chapter 8 is limited to covering the Riemann–Roch theorem and Abel's theorem, which are the most basic theorems regarding analytic functions on compact Riemann surfaces.

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1

Holomorphic functions

1.1 Holomorphic functions

a. The complex plane

An expression $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$, is called a *complex number*. The sum, difference, and product of two complex numbers $z = x + iy$ and $w = u + iv$ are defined by

$$z + w = (x + u) + i(y + v),$$
$$z - w = (x - u) + i(y - v),$$
$$zw = (xu - yv) + i(xv + yu)$$

These expressions are obtained by first evaluating $z + w$, $z - w$, and zw as polynomials in the "variable" i and then replacing i^2 by -1. Therefore, addition, subtraction, and multiplication as defined above satisfy the associative, commutative, and distributive laws.

As usual, the real number line is represented by $\mathbb{R}$. The plane $\mathbb{R}^2$ is the product $\mathbb{R} \times \mathbb{R}$, that is, the collection of all pairs (x, y) of real numbers. If one identifies the point (x, y) of the plane $\mathbb{R}^2$ with the complex number $z = x + iy$, then $\mathbb{R}^2$ is called the *complex plane*. The complex plane is represented by $\mathbb{C}$.

The *absolute value* $|z|$ of the complex number $z = x + iy$ is defined by

$$|z| = \sqrt{x^2 + y^2}$$

For two complex numbers $z = x + iy$ and $w = u + iv$

$$|z - w| = \sqrt{(x - u)^2 + (y - v)^2}$$

is the distance between the points z and w in the plane $\mathbb{C}$. In particular, $|z|$ is the distance between the point z and the origin 0.

If one represents the complex number $z = x + iy$ by the vector $\mathbf{0z}$ from 0 to z, then (x, y) are the coordinates of z and $|z| = \sqrt{(x^2 + y^2)}$ is the length of $\mathbf{0z}$. Therefore, if z_1 and z_2 are complex numbers, and $w = z_1 + z_2$ is their sum, then the vector $\mathbf{0w}$ is equal to the sum of $\mathbf{0z_1}$ and $\mathbf{0z_2}$ (Fig. 1.1):

$$\mathbf{0w} = \mathbf{0z_1} + \mathbf{0z_2}.$$

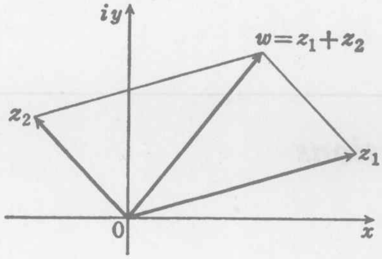

Fig. 1.1

For any complex number $z = x + iy$, one calls $x - iy$ the *conjugate* of z. The conjugate of z is represented by $\bar{z}$:

$$\bar{z} = x - iy.$$

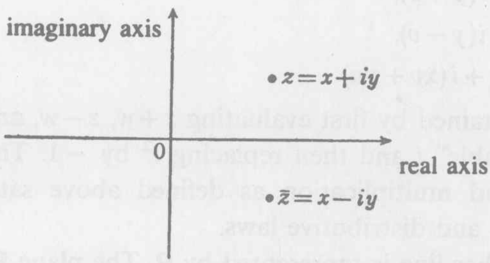

Fig. 1.2

Furthermore, x is called the *real part* of $z = x + iy$, and y is called the *imaginary part*. The real part of z is represented by Re z, the imaginary part by Im z:

$$\operatorname{Re} z = x = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im} z = y = \frac{i(\bar{z} - z)}{2}$$

The line $\mathbb{R} \times \{0\}$ in the complex plane is called the *real axis* and the line $\{0\} \times \mathbb{R}$ is called the *imaginary axis*. The conjugate $\bar{z}$ of z and z are represented by points in the complex plane, that are symmetric with regard to the real axis. Obviously

$$\bar{\bar{z}} = z,$$

$$\overline{z + w} = \bar{z} + \bar{w}, \qquad \overline{z - w} = \bar{z} - \bar{w},$$

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}.$$

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Moreover

$$|z|^2 = |\bar{z}|^2 = x^2 + y^2 = z \cdot \bar{z}.$$

Hence

$$|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2,$$

and therefore

$$|zw| = |z|\,|w|. \tag{1.1}$$

If $z \neq 0$, then $|z| > 0$ and $z \cdot \bar{z}/|z|^2 = 1$. So if $z \neq 0$, then z has an inverse $1/z = \bar{z}/|z|^2$. Therefore, the collection of all complex numbers $\mathbb{C}$ is a field, called the field of complex numbers.

For $z \neq 0$ we have $\overline{(w/z)}\bar{z} = \overline{(w/z)z} = \bar{w}$, therefore

$$\overline{(w/z)} = \bar{w}/\bar{z}.$$

Since similar rules hold for addition, subtraction, and multiplication, as we saw above, it is now clear that if a complex number w is arrived at by a finite number of additions, subtractions, multiplications, and divisions applied to a finite number of complex numbers $z_1, z_2, \ldots, z_n$, then by applying the same operations in the same order to $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ one arrives at $\bar{w}$. Therefore, the correspondence from $\mathbb{C}$ into $\mathbb{C}$ given by $z \to \bar{z}$ is an isomorphism.

For two arbitrary complex numbers, we have the following inequality

$$|z+w| \leqq |z| + |w|. \tag{1.2}$$

Proof: Using $\operatorname{Re} z \leqq |z|$ we have

$$\begin{aligned}|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}\\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \leqq |z|^2 + 2|z\bar{w}| + |w|^2\\ &= |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2.\end{aligned}$$

From the inequality (1.2) the *triangle inequality*

$$|z_1 - z_3| \leqq |z_1 - z_2| + |z_2 - z_3|,$$

where z_1, z_2, z_3 are arbitrary points of the complex plane, follows at once.

From $|z| \leqq |z-w| + |w|$, we conclude $|z| - |w| < |z-w|$.

In the same way it is proved that $|w| - |z| \leqq |w-z|$. Hence

$$\big||z| - |w|\big| \leqq |z-w|. \tag{1.3}$$

Repeated application of (1.1) and (1.2) yields

$$|z_1 z_2 z_3 \cdots z_n| = |z_1|\,|z_2|\,|z_3| \cdots |z_n|,$$
$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|.$$

Therefore

$$|a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n| \leqq |a_0| + |a_1|\,|z| + |a_2|\,|z|^2 + \cdots + |a_n|\,|z|^n.$$

Since the complex plane $\mathbb{C}$ can be identified with the real plane $\mathbb{R}^2$, definitions and theorems pertaining to subsets of $\mathbb{R}^2$ also apply to subsets of $\mathbb{C}$. For example, one says that the sequence $\{z_n\}$ of complex numbers *converges* to w, if the sequence $\{z_n\}$ of points converges to the point w, that is, if

$$\lim_{n\to\infty} |z_n - w| = 0.$$

Theorem 1.1 (*Cauchy's criterion*). The complex sequence $\{z_n\}$ converges if and only if for every real number $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that

$$|z_n - z_m| < \varepsilon \quad \text{if} \quad n > n_0(\varepsilon) \text{ and } m > n_0(\varepsilon).$$

We have $\||z_n| - |w|\| \leqq |z_n - w|$ by (1.3), therefore from $\lim_{n\to\infty} z_n = w$ we can conclude $\lim_{n\to\infty} |z_n| = |w|$. Hence if the complex sequence $\{z_n\}$ converges, then the sequence $\{|z_n|\}$ converges too and we have

$$\lim_{n\to\infty} |z_n| = |\lim_{n\to\infty} z_n|.$$

The infinite series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$ is said to converge if the complex sequence $\{w_n\}$ of partial sums

$$w_n = z_1 + z_2 + \cdots z_n$$

converges. The complex number $w = \lim_{n\to\infty} w_n$ is called the *sum* of the series and we write

$$w = \sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots.$$

If the sequence $\{w_n\}$ does not converge, then the series $\sum_{n=1}^{\infty} z_n$ is called *divergent*.

Putting $\sigma_n = |z_1| + |z_2| + \cdots + |z_n|$, we have for $m < n$

$$|w_n - w_m| = \left| \sum_{k=m+1}^{n} z_k \right| \leqq \sum_{k=m+1} |z_k| = \sigma_n - \sigma_m.$$

Applying Cauchy's criterion we conclude that $\sum_{n=1}^{\infty} z_n$ converges if $\sum_{n=1}^{\infty} |z_n|$ converges. In this case, $\sum_{n=1}^{\infty} z_n$ is called *absolutely convergent*.

If $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} z_n\right| = \left|\lim_{m\to\infty} w_m\right| = \lim_{m\to\infty} |w_m| \leqq \lim_{m\to\infty} \sum_{n=1}^{m} |z_n| = \sum_{n=1}^{\infty} |z_n|.$$

Since $\sum_{n=1}^{\infty} |z_n|$ either converges or diverges to $+\infty$, $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} |z_n| < +\infty$. If $w = \sum_{n=1}^{\infty} z_n$ and $\omega = \sum_{n=1}^{\infty} \zeta_n$ are both absolutely convergent, then

$$\begin{aligned} w\cdot\omega = {} & z_1\zeta_1 + z_2\zeta_1 + z_1\zeta_2 + z_3\zeta_1 \\ & + z_2\zeta_2 + z_3\zeta_1 + z_2\zeta_2 + z_1\zeta_3 + \cdots . \end{aligned} \tag{1.4}$$

Proof: Putting

$$\sigma_n = |z_n|\,|\zeta_1| + |z_{n-1}|\,|\zeta_2| + |z_{n-2}|\,|\zeta_3| + \cdots + |z_1|\,|\zeta_n|$$

we have $\sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty} |z_n| \sum_{n=1}^{\infty} |\zeta_n|$, so that the series of the right-hand side of (1.4) is absolutely convergent and

$$\begin{aligned} & \left|\sum_{n=1}^{m} z_n \sum_{n=1}^{m} \zeta_n - \sum_{n=1}^{m} (z_n\zeta_1 + z_{n-1}\zeta_2 + \cdots + z_1\zeta_n)\right| \\ & \qquad \leqq \sum_{n=1}^{m} |z_n| \sum_{n=1}^{m} |\zeta_n| - \sum_{n=1}^{m} \sigma_n \to 0, \qquad m\to\infty. \end{aligned}$$

b. Functions of a complex variable

Let D be a subset of $\mathbb{C}$, i.e., D is a point-set in the complex plane. A function f defined in D assigns to each element of D exactly one complex number. D is called the *domain* of f. For $\zeta \in D$ the complex number ω assigned to ζ by f is called the *value* of f at ζ. We write

$$\omega = f(\zeta).$$

If S is a subset of D, then the collection of all complex numbers $f(\zeta)$, where $\zeta \in S$, is written as $f(S)$

$$f(S) = \{f(\zeta): \zeta \in S\}.$$

The set $f(D)$ of all values $\omega = f(\zeta)$ is called the *range* of f.

Writing $f(z)$ instead of f, one calls z a *variable* and $f(z)$ a *function of a complex variable*. Just as for functions of a real variable, z denotes an arbitrary element ζ of D, or, in other words, a symbol for which ζ has to be substituted. According to general custom, we will use the same letter z to denote points of D. Putting $w = f(z)$, we call w a function of z.

An open subset U of the complex plane is said to be *connected* if U is not the union of two nonempty, open subsets that have no points in common. An open subset U is connected if and only if each pair of points z, w of U can be connected by an arc lying in U.

A connected open subset of $\mathbb{C}$ is called a *region* (or a *domain*); the closure of a region is called a *closed region* (or a *closed domain*).

In this book we will mainly consider functions defined on regions or on closed regions, but we start by discussing limits, continuity, and other properties of functions, defined on arbitrary sets $D \subset \mathbb{C}$.

Definition 1.1. Let D be a point set in $\mathbb{C}$, c an accumulation point of D, and γ a complex number. We say that $f(z)$ *converges* to γ or that γ is the *limit* of $f(z)$ as z tends to c, if for every real $\varepsilon > 0$ there exists a real $\delta(\varepsilon) > 0$ satisfying

$$|f(z) - \gamma| < \varepsilon \qquad \text{if } 0 < |z - c| < \delta(\varepsilon). \tag{1.5}$$

This is written as

$$\lim_{z \to c} f(z) = \gamma$$

or

$$f(z) \to \gamma \quad \text{as } z \to c.$$

Since $f(z)$ is not defined if $z \notin D$, we have to assume that $z \in D$ in (1.5). The assumption that c is an accumulation point of D is necessary to exclude the possibility that there are no points z satisfying $z \in D$ and $0 < |z - c| < \delta(\varepsilon)$.

The proof of the following result is similar to the proof of the corresponding result for real functions.

The function $f(z)$ converges to γ as z tends to c if and only if for all complex sequences $\{z_n\}$, $z_n \in D$ and $z_n \neq c$, converging to c the complex sequence $\{f(z_n)\}$ converges to γ.

Combining this theorem with Cauchy's criterion for complex sequences, we arrive at Cauchy's criterion for functions.

Theorem 1.2 (Cauchy's criterion). Let $f(z)$ be a function of the complex variable z defined on $D \subset \mathbb{C}$ and let c be an accumulation point of D. Then $f(z)$ converges to some value if z tends to c if and only if for every real $\varepsilon > 0$ there exists a real $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(w)| < \varepsilon \quad \text{if} \quad 0 < |z - c| < \delta(\varepsilon) \text{ and } 0 < |w - c| < \delta(\varepsilon).$$

Let $f(z)$ be a complex function defined on $D \subset \mathbb{C}$, and assume that c belongs to D. If

$$\lim_{z \to c} f(z) = f(c), \tag{1.6}$$

then $f(z)$ is said to be *continuous* at c.

It follows at once from the definition that $f(z)$ is continuous at c if and only if for every real $\varepsilon > 0$ there exists a real $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(c)| < \varepsilon \quad \text{if} \quad |z - c| < \delta(\varepsilon).$$

(If c is an isolated point of D, then for sufficiently small δ the only z satisfying $z \in D$ and $|z - c| < \delta(\varepsilon)$ is c, in which case $f(z)$ is certainly continuous at c.)

Putting $z = x + iy$ and $c = a + ib$, we can split $f(z)$ into a real and an imaginary part

$$f(z) = u(z) + iv(z), \quad u(z) = \operatorname{Re} f(z), \quad v(z) = \operatorname{Im} f(z).$$

The real part and the imaginary part can be considered as real functions of two real variables x and y where $z = x + iy$. From

$$|f(z) - f(c)| = \sqrt{|u(x, y) - u(a, b)|^2 + |v(x, y) - v(a, b)|^2}$$

we conclude that (1.6) is equivalent to

$$\lim_{(x, y) \to (a, b)} u(x, y) = u(a, b), \qquad \lim_{(x, y) \to (a, b)} v(x, y) = v(a, b).$$

Therefore, the function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ is continuous at $c = a + ib$ if and only if its real part $u(x, y)$ and its imaginary part $v(x, y)$ are continuous at (a, b) as functions of the two real variables x and y.

If the complex function $f(z)$ is continuous at all points of its domain $D \subset \mathbb{C}$, then f is called a continuous function of z or simply a *continuous function*. The function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ is continuous if and only if its real part $u(x, y)$ and its imaginary part $v(x, y)$ are continuous functions of the two real variables x and y.

Just as for functions of a real variable, limits of complex functions satisfy the following rules: let $f(z)$ and $g(z)$ be functions of a complex variable z defined on $D \subset \mathbb{C}$ and let c be an accumulation point of D. If both $f(z)$ and $g(z)$ converge to a limit as $z \to c$ then the linear combination $a_1 f(z) + a_2 f(z)$, where a_1 and a_2 are constants and the product $f(z) \cdot g(z)$ converge to a limit and these limits satisfy

$$\lim_{z \to c} (a_1 f(z) + a_2 g(z)) = a_1 \lim_{z \to c} f(z) + a_2 \lim_{z \to c} g(z),$$

$$\lim_{z \to c} f(z) g(z) = \lim_{z \to c} f(z) \cdot \lim_{z \to c} g(z).$$

If moreover $\lim_{z \to c} g(z) \neq 0$, then the quotient $f(z)/g(z)$ converges if $z \to c$ and the limit satisfies

$$\lim_{z \to c} \frac{f(z)}{g(z)} = \frac{\lim_{z \to c} f(z)}{\lim_{z \to c} g(z)}.$$

Hence if $f(z)$ and $g(z)$ are continuous functions of z, then the linear combination $a_1 f(z) + a_2 g(z)$ and the product $f(z)g(z)$ are continuous functions. If, moreover, $g(z) \neq 0$ for all $z \in D$, then the quotient $f(z)/g(z)$ is also a continuous function of z.

Continuity of the composite of two complex functions obeys the same rule as real functions do: If $f(z)$ is a continuous function of z defined on $D \subset \mathbb{C}$, if $g(w)$ is a continuous function of w defined on $E \subset \mathbb{C}$ and if $f(D) \subset E$, then the composite $g(f(z))$ is a continuous function of z on D. For, if c is an arbitrary point of D, then $\lim_{z \to c} f(z) = f(c)$ and $\lim_{w \to f(c)} g(w) = g(f(c))$; hence $\lim_{z \to c} g(f(z)) = g(f(c))$.

The functions z and $\bar{z}$ are obviously continuous functions of z defined on $\mathbb{C}$. According to the above, linear combinations of finite products of z and $\bar{z}$, that is polynomials in z and $\bar{z}$:

$$f(z) = \sum_{h=0}^{m} \sum_{k=0}^{n} a_{hk} z^g \bar{z}^k, \qquad a_{hk} \in \mathbb{C}$$

are continuous functions of z.

Definition 1.2. Let $f(z)$ be a continuous function of z defined on $D \subset \mathbb{C}$. If for every real $\varepsilon > 0$ there exists a real $\delta(\varepsilon) > 0$ such that

$$|f(z) - f(w)| < \varepsilon \qquad \text{if } |z - w| < \delta(\varepsilon) \text{ and } z \in D \text{ and } w \in D$$

then $f(z)$ is said to be *uniformly continuous* on D.

Theorem 1.3. A continuous function $f(z)$ defined on a bounded, closed set $D \subset \mathbb{C}$ is uniformly continuous on D.

Proof: Assume that $f(z)$ is not uniformly continuous on D. Then there exists an $\varepsilon > 0$, such that for each δ it is not true that $|f(z) - f(w)| > \varepsilon$ whenever $|z - w| < \delta$, $z \in D$, and $w \in D$. Hence there exist complex numbers z_n and w_n for each natural number n, satisfying

$$|z_n - w_n| < \frac{1}{n}, \qquad z_n \in D, \qquad w_n \in D, \qquad |f(z_n) - f(w_n)| \geqq \varepsilon. \quad (1.7)$$

Since D is bounded, there exists a subsequence $z_{n_1}, z_{n_2}, \ldots, z_{n_j}, \ldots, n_1 < n_2 < \cdots < n_j < \cdots$, of the complex sequence $\{z_n\}$, which converges.

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