DIFFUSIONS, MARKOV PROCESSES AND MARTINGALES

Volume 2
Itô Calculus

扩散 马尔可夫过程和鞅
第 2 卷

L. C. G. Rogers & D. Williams

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Diffusions, Markov Processes, and Martingales

Volume 2: ITÔ CALCULUS

2nd Edition

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Preface

(*a*) *Welcome back on board!* You will have noticed that for this second leg of your journey, there are two pilots rather than one. D.W. is sure that you will be as delighted as he is that control is being shared with L.C.G.R.—amongst so many other things, just the man for a Wiley excursion!

We apologize for the considerable delay in departure. Anyone who knows what has been happening to British universities will need no further explanation, and will share our sadness.

(*b*) The book is meant to help the research student reach the stage where he or she can begin both to think up and tackle new problems and to read the up-to-date literature across a wide spectrum; and to persuade him or her that it is worth the effort.

We can say that we ourselves find the subject sufficiently good fun to have enjoyed the task of writing. (We even had some amusement from typing the manuscript ourselves with the very basic non-mathematical word-processor VIEW on the BBC micro. Occasionally, we got into trouble when trying to use global editing to substitute the most commonly occurring phrases for shorthand versions of our own devising. But, in the main, we were very satito's formulaied!)

(*c*) Chapter IV, *Introduction to Itô calculus*, is particularly concerned with developing the theory of the stochastic integral (of a previsible process) with respect to a continuous semimartingale, and with giving a large number of applications. Chung and (Ruth) Williams [1] would make a splendid companion volume for this chapter.

Chapter V, *Stochastic differential equations and diffusions*, presents first the theory of SDEs: existence and uniqueness for strong and weak solutions, martingale problems, etc. It has an extended treatment of 1-dimensional diffusions, and a huge attempt to introduce the very fashionable subject of stochastic differential geometry. Strongly recommended 'parallel' reading for this chapter: McKean's sparkling book [1] and the authoritative Ikeda and Watanabe [1].

Chapter VI, *The general theory*, presents *la théorie générale*: dual previsible projections, the Meyer decomposition theorem, the general integral, etc., with a chunky piece on excursions. The literature on the general theory is dominated by the masterly account by Dellacherie and Meyer ([1]) who created so much of it. Dellacherie's own very fine survey article [3], Jacod [2], Metivier and Pellaumail [1] should also be consulted.

In everything, the Russian literature, as represented by such important volumes as Gikhman and Skorokhod [1] and Liptser and Shiryayev [1], has its own characteristic style and special value.

(*d*) The book has a large bibliography, but this represents a small and rather haphazard selection of what we should have included. We apologize for the enormous number of very important papers which are omitted.

Numerous important topics are omitted too, or given treatment far too brief for their true significance. (Reviewers who find the previous sentence handy are free to use it without acknowledgement.) So, here are some guidelines on what you might move on to when your reading of our book is done.

(*e*) (i) *Large deviations.* The recent appearance of books by Stroock [4] and the grandmaster himself, Varadhan [1], would have made any efforts from us look silly. This is the only reason for our omission of this topic and for the (otherwise scandalous) omission from the bibliography of the historic papers by Donsker and Varadhan and by Ventcel and Freidlin.

(ii) *Malliavin calculus.* See § V.36.

(iii) *Large deviations and Malliavin calculus.* See Bismut [4], and also Elworthy and Truman [1, 2] for important work which provided motivation. Keep a look out for forthcoming work by Léandre.

(iv) *Markov processes.* The value of the classics mentioned in Volume 1—Blumenthal and Getoor [1], Getoor [1], and Meyer [3]—remains as great as ever. Sharpe [1] is sure to be a definitive account, as (of course) is that provided by later volumes of Dellacherie and Meyer [1]. (Volume 4 of the latter has arrived just as we are posting off the final proofs. Splendid to look forward to reading it!). The volumes in the 'Seminars on stochastic processes' (Çinlar, Chung and Getoor [1]) are important state-of-the-art reports.

Ethier and Kurtz [1] is a valuable source for much theory, for the establishment of weak-convergence results, etc. Liggett's account [1] of one of the most important application areas, interacting particle systems, is magisterial.

For a profound study of the relationship between Markov processes and semimartingales, see Çinlar, Jacod, Protter and Sharpe [1].

For applications to potential theory and complex analysis, see Doob [3], Durrett [1], and Port and Stone [3]. Two papers by Lyons [1, 2] are very much recommended.

(v) *Quantum theory.* So much has been achieved in interrelating quantum theory and probability that one hardly knows where to begin, but an excellent lead-in is provided by de Witte-Morette and Elworthy [1].

It is essential to realize that some of the finest work on probability is being done by people who are first and foremost mathematical physicists or functional analysts. See Simon [1, 2], Davies and Simon [1], Aizenmann and Simon [1], and the literature you can trace through them.

Local time and self-intersection local time have come to play a big part in the construction of quantum fields. See Geman and Horowitz [1], Rosen [1], Geman, Horowitz and Rosen [1], Le Gall [2, 3], Yor [4, 5] and then Dynkin [5, 6, 7] to begin your study in this area.

Whatever the philosophical problems, Nelson's *stochastic mechanics* is certainly prompting very interesting mathematics. See Nelson [1, 2] and Carlen [1, 2].

A fascinating theory of *non-commutative stochastic integrals* and of non-commutative SDEs has been created by Hudson and Parthasarathy. Meyer [1, 2] is a splendid attempt to make probabilists informed and involved.

(vi) *Measure-valued diffusions, random media, etc.*. Durrett [2] and Dawson and Gärtner [1] can be your 'open sesame' to what is sure to be one of the richest of Aladdin's caves.

(vii) *The Séminaires.* It is impossible to overstate our indebtedness to the famous *Séminaires de Probabilités*, originated by Meyer and developed by him (with help from Dellacherie and Weil) into an absolutely indispensable handbook, and now maintained as such in Azéma and Yor's expert hands. *Séminaire* XX: *Springer Lecture Notes in Mathematics Volume* 1204, contains an index to the series so far.

(*f*) *Further acknowledgements.* The work on this book has been done at the Universities of Wales (Swansea), Warwick and Cambridge, all of which deserve our thanks.

Most was done at Swansea where both of us spent very happy times. Special thanks to Aubrey Truman, Peter Townsend and Betty Williams.

We thank our colleagues at Cambridge for their warm welcome; and are pleased to acknowledge the help and advice we have received from many, especially Frank Adams, Keith Carne, David Kendall and James Norris.

Our best thanks to Sheila Williams, amanuensis extraordinary, who is just about to rediscover after a long period that there are such things as a dining-room table and a sideboard in the Williams household.

And, of course, our thanks to Charlotte Farmer, Robert Hambrook and the other staff of Wiley for making sure that it has become a reality; and to copy editors, and to wonderfully accurate typesetters.

Cambridge, October 1986

Chris Rogers
David Williams

Added, April 2000

Our thanks too to the staff of C.U.P., especially David Tranah, and also to the wonderfully accurate typesetters for their superb 'invisible mending'.

Contents

CHAPTER V. STOCHASTIC DIFFERENTIAL EQUATIONS AND DIFFUSIONS

CHAPTER IV

Introduction to Itô Calculus

Here, we give the gist of the 'martingale and stochastic integral' method, and illustrate its use via a large number of fully-worked examples. We do not apologize for sometimes advertising the method by showing how it can obtain results which are well known and elementary. Thus, for example, we take the trouble to prove some standard results about the humble Markov chain with finite state-space. But we have also tried to bring into this chapter applications which are less elementary, and which hint at the excitement of the subject today.

TERMINOLOGY AND CONVENTIONS

R-processes and L-processes

We now use the term *R-process* on $[0, \infty)$ to signify a process all of whose paths are right-continuous on $[0, \infty)$ with limits from the left on $(0, \infty)$. Thus an R-process is what was called in Volume 1 a Skorokhod process, and what is called elsewhere a càdlàg process, or a corlol process, or whatever. An R-function or R-path on $[0, \infty)$ is defined via the obvious analogous definition.

The *L-processes* on $(0, \infty)$, all of whose paths are left-continuous with limits from the right, will now begin to feature largely in the theory.

Usual conditions, etc.

Everywhere in this chapter, we work with a set-up $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P})$ *satisfying the 'usual conditions'. See (II.67).*

All martingales (and 'finite-variation processes', and 'semimartingales') will be taken to be R-processes. Because we are assuming that the usual conditions hold, this is in order. See (II.67).

We shall also always assume that a process $\{X_t: t \geqslant 0\}$ *is jointly measurable; that is, the map* $(t, \omega) \mapsto X_t(\omega)$ *is measurable with respect to* $\mathscr{B}(\mathbb{R}^+) \times \mathscr{F}$.

Recall that the process X is said to be adapted if X_t *is* $\mathscr{F}_t$*-measurable for every* $t \geqslant 0$.

Important convention about time 0

Our stochastic integrals will be defined over intervals $(0, t]$ open at 0. Thus, the value of the integral at time 0 will be 0. This differs from the convention in Dellacherie and Meyer [1]. As explained there, time 0 plays *le rôle du diable*. We consign it to Hell.

In accordance with this convention, the parameter set for our previsible processes will be the open interval $(0, \infty)$.

1. SOME MOTIVATING REMARKS

1. Itô integrals. One of our main tasks is to define the Itô integral

$$\int H\,dX,$$

where H and X are stochastic processes of appropriate classes.

We shall regard this integral as a new stochastic process, often written $H \bullet X$, and shall use the alternative notations:

$$(H \bullet X)(t, \omega) = \left(\int_{(0,t]} H_s\,dX_s \right)(\omega). \tag{1.1}$$

We shall often use differential notation, in which we can rewrite equation (1.1) as a 'stochastic differential equation':

$$d(H \bullet X) = H\,dX. \tag{1.2}$$

The theory is now essentially complete in the sense that it is known exactly what conditions need to be imposed on our integrand H and integrator X:

The essential requirement on the integrand H is that it be 'previsible'.
The integrator X must be a 'semimartingale'.

The most important example of a previsible process is provided by an *adapted L-process*. Indeed, the adapted L-processes 'generate' the previsible processes, as will be explained later. Let H be an adapted L-process. Then H_s is known to the observer at time s. The reason that H is 'previsible' is (roughly speaking) that, for a stopping time $T > 0$, H_T is known immediately before time T because

$$H_T = \lim_{s \uparrow\uparrow T} H_s.$$

The simplest adapted L-process is the process

$$H = 1(S, T], \tag{1.3}$$

where S and T are stopping times with $S < T$. Thus,

$$H(t, \omega) = \begin{cases} 1 & \text{if } S(\omega) < t \leqslant T(\omega), \\ 0 & \text{otherwise.} \end{cases}$$

Let M be a martingale. Then, for H as in equation (1.3), the obvious definition of $H \bullet M$ is:

$$(H \bullet M)_t = M_{T \wedge t} - M_{S \wedge t},$$

and we can easily show that *$H \bullet M$ is a martingale.*

From this simple case develops *the most fundamental property of Itô integrals*:

(*1.4*) ITÔ INTEGRALS PRESERVE LOCAL MARTINGALES. It will do no harm to give a precise statement of this now. The reader new to the subject will not know what Theorem 1.5 means, but it will help him or her to know that it is one of the main landmarks in our route through the subject.

(*1.5*) **FUNDAMENTAL THEOREM.** *If H is a locally bounded previsible process and M is a local martingale, then $H \bullet M$ exists and is a local martingale.*

We could very easily explain now what a locally bounded previsible process is. We could also easily explain what a local martingale is; indeed, let us do it:

(*1.6*) DEFINITION (local martingale): *A process M is a* local martingale *if M_0 is $\mathcal{F}_0$ measurable and there exists an increasing sequence of stopping times (T_n) with $T_n \uparrow \infty$ such that each 'stopped' process*

$$\{M_{T_n \wedge t} - M_0 \colon t \geqslant 0\}$$

is a martingale.

What we cannot explain in a short space is what $H \bullet M$ means in the generality of Theorem 1.5. But the discrete-time setting explains why Theorem 1.5 is true.

(*1.7*) *A discrete-time analogue.* Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathrm{P})$ be a discrete-time set-up, and let M be an associated martingale. Let H be a bounded process previsible in the sense that

$$Z_{n-1} = H_n \in \mathrm{b}\mathcal{F}_{n-1} \quad (n \in \mathbb{N}).$$

Define:

$$(H \bullet M)_n = \sum_{k=1}^{n} H_k (M_k - M_{k-1}) = \sum_{k=1}^{n} Z_{k-1}(M_k - M_{k-1})$$

$$(H \bullet M)_0 = 0.$$

Then $H \bullet M$ is a martingale.

Proof. To show that a process N is a martingale, we need only show that

$$\mathbf{E}[N_n - N_{n-1} | \mathscr{F}_{n-1}] = 0, \quad n \in \mathbb{N}.$$

But, since $Z_{n-1} \in b\mathscr{F}_{n-1}$,

$$\begin{aligned}\mathbf{E}[(H \bullet M)_n - (H \bullet M)_{n-1} | \mathscr{F}_{n-1}] &= \mathbf{E}[Z_{n-1}(M_n - M_{n-1}) | \mathscr{F}_{n-1}] \\ &= Z_{n-1}\mathbf{E}[M_n - M_{n-1} | \mathscr{F}_{n-1}] = 0.\end{aligned}$$

□

As was mentioned earlier, *the general 'integrator' X will be a semimartingale.* This means that X *may be written in the form*

(1.8) $$X = X_0 + M + A,$$

where X_0 is $\mathscr{F}_0$ measurable, M is a local martingale null at 0, *and A is an adapted process with paths of finite variation, also null at* 0.

In this chapter, we present the full theory for two special cases of great importance:

(i) *the case in which $X = A$, a process with paths of finite variation*;
(ii) *the case in which the paths of X are continuous.*

This will allow us to develop many of the main applications. The general theory is given in Chapter VI.

If A is a process with paths of finite variation, then (for a bounded measurable process H) we can define $H \bullet A$ as the Stieltjes integral for each ω:

$$(H \bullet A)(t, \omega) = \int_{(0,t]} H(s, \omega)\, dA(s, \omega).$$

Though no new concept of integration is involved here, the theory is extremely useful because of what Theorem 1.5 says in this context:

(*1.9*) THEOREM. *Let H be a locally bounded previsible process, and let M be a local martingale with paths of finite variation. Then $H \bullet M$, as defined by the Stieltjes integral, is a local martingale.*

If M is a (path-) continuous local martingale, then the paths of M generally will not have finite variation. Indeed, the only paths of finite variation will be constant! Thus the integral $H \bullet M$ (where H is a locally bounded previsible process) is a true extension of the Stieltjes integral. The very existence of the integral is inextricably tied up with its calculus, that is, with the *integration-by-parts* formula and the *pièce de résistance* of the theory, *Itô's formula.*

2. Integration by parts. The most important integral associated with a local martingale M is the integral $\int_{(0,t]} M_{s-} dM_s$. The adapted L-process $M_- = \{M_{s-}: s > 0\}$ is previsible and also locally bounded, so the integral exists. More generally, if X and Y are semimartingales, then the Itô integral $\int X_{s-} dY_s$ may be defined. Now the integral $\int X_{s-} dY_s$ is analogous to a sum of the form

(2.1) $$\sum x_{k-1}(y_k - y_{k-1}).$$

The summation-by-parts formula for such sums:

(2.2) $$x_n y_n - x_0 y_0 = \sum_{k=1}^{n} x_{k-1}(y_k - y_{k-1}) + \sum_{k=1}^{n} y_{k-1}(x_k - x_{k-1}) + \sum_{k=1}^{n} (x_k - x_{k-1})(y_k - y_{k-1})$$

suggests the fundamental *integration-by-parts formula for semimartingales*:

(2.3) $$X_t Y_t - X_0 Y_0 = \int_{(0,t]} X_{s-}\, dY_s + \int_{(0,t]} Y_{s-}\, dX_s + \int_{(0,t]} dX_s\, dY_s.$$

But what sense are we to make of the last term in (2.3)?

It is easy to believe (and to prove!) that if X and Y have paths of finite variation, then the correct interpretation is as follows:

(2.4) $$\int_{(0,t]} dX_s\, dY_s = \sum_{0<s\leqslant t} (X_s - X_{s-})(Y_s - Y_{s-}).$$

What happens if X and Y are path-continuous martingales? To gain insight into this situation, and into more general situations, we again look at a discrete analogue.

(2.5) *A discrete-time analogue.* Let $M = \{M_n\colon n \geqslant 0\}$ and $N = \{N_n\colon n \geqslant 0\}$ be martingales on a set-up $(\Omega, \mathscr{F}, \{\mathscr{F}_n\}, \mathbf{P})$ such that for all n, $\mathbf{E}M_n^2 < \infty$, $\mathbf{E}N_n^2 < \infty$. Take $x_k = M_k(\omega)$, $y_k = N_k(\omega)$ in (2.2) to see that

$$M_n N_n - M_0 N_0 = \sum_{k=1}^{n} M_{k-1}(N_k - N_{k-1}) + \sum_{k=1}^{n} N_{k-1}(M_k - M_{k-1}) + \sum_{k=1}^{n} \Delta M_k \Delta N_k,$$

where $\Delta M_k = M_k - M_{k-1}$ and $\Delta N_k = N_k - N_{k-1}$. Now (compare (1.6)):

$$U_n = \sum_{k=1}^{n} M_{k-1}(N_k - N_{k-1})$$

defines a martingale U because

$$\mathbf{E}[U_n - U_{n-1} \mid \mathscr{F}_{n-1}] = M_{n-1}\mathbf{E}[N_n - N_{n-1} \mid \mathscr{F}_{n-1}] = 0.$$

Hence, if we put

$$[M, N]_n = \sum_{k=1}^{n} \Delta M_k \Delta N_k,$$

then

$M_n N_n - M_0 N_0 - [M, N]_n$ *is a martingale.*