21 世纪高等院校教材

Equations of Classical Mathematical Physics

(经典数学物理方程)

谢鸿政 编

科学出版社
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内 容 简 介

本书是数学物理方程课程的英文教材，共 10 章，内容包括：绪论、数学模型与定解问题、二阶线性偏微分方程的分类和化简、特征线积分法、分离变量法、本征值问题与特殊函数、高维边值问题、积分变换法、调和函数的基本性质、格林函数及其应用等.

本书可作为高等学校理工科(非数学专业)本科生和研究生的公共专业或技术基础课英文教材，也可供科技工作者参考.

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编　者

2006 年 1 月

Preface

Equations of Mathematical Physics, as an important specialized basic course of universities (or colleges), has close relation with the modern science and engineering technique, and has wide use as the applying mathematical basis.

Besides showing basic concepts and principles on partial differential equations (brief$\sim$PDEs), the focus of attention in this book is discussing main methods and techniques for solving basic definite problems.

The contents of this book are developed as follows. The first chapter is concerned with basic concepts and definitions on partial differential equations.

In Chapter 2, typical mathematical models (string oscillation, heat conduction and Laplace equations) describing basic physical phenomena and basic problems for defining solutions are showed.

In Chapter 3, the classification, simplification and typical forms for linear second order PDEs in two independent variables are demonstrated.

In Chapter 4, the integral method on characteristics for solving hyperbolic and parabolic equations are showed.

In Chapter 5, the method of separation of variables for solving the problems with boundary conditions on finite regions are discussed.

In Chapter 6, the eigenvalue problems and corresponding Sturm-Liouville problem as well as special functions are presented. In addition, the Green function method for solving boundary value problems of linear second order ordinary differential equations is showed.

In Chapter 7, some important examples, as applications of separation of variables for solving multidimensional problems with complicated boundary conditions, are described.

In Chapter 8, the definition, basic properties on Fourier and Laplace integral transformations and some applications for solving the definite problems in unbounded region for PDEs are showed.

In Chapter 9, the definition, basic properties of the harmonic function and some applications concerning boundary value problems for Laplace equation are presented.

In Chapter 10, the definition, basic properties of Green functions and important applications for solving Dirichlet and Neumann problems of the equations with Laplace operator are showed.

Hongzheng Xie

2006.1

Contents

Chapter 1
Introduction

1.1 Equations of mathematical physics

Mathematical physics is an inter-relating science. On the basis of the fundamental laws of physics, mathematical methods are used to study processes evolving in material media. Its purpose is to formulate equations describing a process within a reasonable degree of idealization (i.e., disregarding details that are not essential for its qualitative and quantitative essences), to develop methods for solution of the resulting problem, and to analyse the qualitative and quantitative properties of the solutions. In this latter respect mathematical physics borders on numerical analysis and mathematical simulation, but in its most important aspects it borders on the theoretical and even experimental natural science.

We shall restrict our attention to phenomena of the "macro" world-more precisely, to processes evolving in continuous media. At this point some elaboration of the very notion of a continuous medium is desirable, since at first sight it might seem incompatible with an atomistic view of the universe. The notion of continuous medium is related to the following notion of a physical element of volume. Consider some process evolving in a region $D \subset \mathbf{R}^3$ and let $K \subset D$ be a subset of positive three-dimensional measure with diameter d:

$$d = \max_{p \in K} \max_{q \in K} r_{pq}, \tag{1.1.1}$$

where r_{pq} is the distance between points p and q. Fixing $p \in K$, consider it as a representative of K. Assume that d is much smaller than the characteristic size of D (e.g., the upper limit of the diameters of all spheres contained in D), but that the number of individual material particles in K is very large and their maximum size is very small compared with d. Consider some physical characteristic F of particles in K (e.g., the velocity V at time t of the individual particles in K). Let $\hat{F}(p,t)$ denote the value of F averaged over all particles in K. The medium in D may be called continuous with respect to F if $\hat{F}(p,t)$ is a continuous function of p and t everywhere in D, except for finitely many sets of points of zero three-dimensional measure (i.e., except for finitely many surfaces, lines, or separate points). If the medium is continuous with respect to all physical parameters of relevance for the process in question, one can speak of the medium as simply continuous.

Processes in nature may be divided, roughly speaking, into three groups: (1) stationary processes, in which the state of the system is independent of time; (2) dissipative time-dependent evolution processes; (3) conservative evolution processes.

There is a similarity among the fundamental laws governing processes of the same group. For examples, Fourier's law of heat conduction, Fick's law of diffusion, and Darcy's law of liquid percolation through porous media are identical, up to renaming of the variables. Indeed, Fourier's law read: the amount of heat flowing in an isotropic homogeneous thermally conductive body through a surface element $d\tilde{\sigma}$ in the direction of the normal $\boldsymbol{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\lambda \frac{\partial}{\partial \boldsymbol{n}} \tilde{T} d\tilde{\sigma} d\tilde{t}, \tag{1.1.2}$$

where $\tilde{T}$ is the temperature and the minus sign indicates that the heat is flowing in the direction of decreasing temperature, so the coefficient $\tilde{\lambda}$ of thermal conductivity may be assumed positive.

Now Fick's law reads that the mass of a solute transferred by diffusion in an isotropic solution through a surface element $d\sigma$ in the direction of the normal $\boldsymbol{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\tilde{D} \frac{\partial}{\partial \boldsymbol{n}} \tilde{C} d\tilde{\sigma} d\tilde{t}, \tag{1.1.3}$$

where $\tilde{C}$ is the solute concentration.

Finally, Darcy's law reads that the mass of liquid percolating through the pore space of a homogeneous porous medium through a surface element $d\tilde{\sigma}$ in direction of the normal $\boldsymbol{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\tilde{K}\frac{\partial}{\partial \boldsymbol{n}}\tilde{p}d\tilde{\sigma}d\tilde{t}, \tag{1.1.4}$$

where $\tilde{p}$ is the pore pressure and $\tilde{K}$ is the percolation coefficient. The tilde "$\sim$" indicates that variables are dimensional.

All these phenomenological laws are of the same form, written in terms of dimensionless variables, and they are indistinguishable. It is this possibility of simultaneously describing processes of a different physical nature, but belonging to the same group, that makes mathematical physics a universal language of the continuum, a connecting link between different disciplines of physics, chemistry, biology, and so on. The interrelation between the various properties of partial differential equations and their natural prototypes is very profound and helpful for research. Quite frequently, previously unknown mathematical phenomena are discovered by looking for the explanation of a natural phenomenon and vice versa, a natural phenomenon may be predicted by analyzing the properties of the corresponding mathematical object.

In general, the equations of mathematical physics include partial differential equations (PDEs), ordinary differential equations (ODEs), integral equations and integral-differential equations which are presented from Physics, Mechanics, Astronomy, Chemistry, Biology and Engineering. However, PDEs are main contents and also main topics of our study.

1.2 Basic concept and definition

Partial differential equation

Typical form:

$$f(x_1, x_2, \cdots, u, u_{x_1}, u_{x_2}, \cdots, u_{x_1x_1}, u_{x_1x_2}, \cdots) = 0, \tag{*}$$

where

$x_1, x_2, \cdots$—independent variables;

$u = u(x_1, x_2, \cdots)$—unknown function of independent variables;

$u_{x_1}, u_{x_2}, \cdots u_{x_1x_1}, u_{x_1x_2}, \cdots$—partial derivatives of unknown function u on independent variables $x_1, x_2, \cdots$ and $x_1x_2 \cdots \in D \subset \mathbf{R}^n$, $n \geqslant 2$.

where $\mathbf{R}^n$—n-dimensional Euclidean space; D—open region in $\mathbf{R}^n$.

The solution of equation (*)

If there exists a sufficiently smooth function $u = u(x_1, x_2, \cdots)$(i.e., there are various continuous partial derivatives in D which appear in the equation (*) and $u(x_1, x_2, \cdots)$ is continuous on $\bar{D}$), such that $u(x_1, x_2, \cdots)$ satisfies the equation (*) in D, then the $u(x_1, x_2, \cdots)$ is called the solution of the equation (*).

Examples. Partial differential equations:

$uu_{xy} + u_x = y,$
$u_{xx} + 2yu_{xy} + 3xu_{yy} = 4\sin x,$
$(u_x)^2 + (u_y)^2 = 1,$
$u_{xx} - u_{yy} = 0.$

It is easy to verify that the two functions $u(x, y) = (x + y)^3, u(x, y) = \sin(x - y)$ are the solutions of the equation $u_{xx} - u_{yy} = 0$.

Order of partial differential equation

— the highest order among all orders of partial derivatives of unknown function u in equation (*).

Examples. $u_{xx} + 2uu_{xy} + u_{yy} = e^y$ is a second order equation, and $u_{xxy} + xu_{yy} + 8u = 7y$ is an equation of third order.

Linear equation. All unknown functions and their partial derivatives are linear and all coefficients in the equation only depend on independent variables.

Quasilinear equation. All the partial derivatives of the highest orders are linear but the equation is not linear.

Nonlinear equation. The partial derivatives of the highest order are nonlinear.

Examples. $yu_{xx} + 2xyu_{xy} + u_y = x^2$ is a second order linear equation; $u_xu_{xx} + xuu_y = \cos x$ is a second order quasilinear equation; $(u_{xy})^2 + 5u_x + e^yu = y^2$ is a second order nonlinear equation.

General form. Second order linear partial differential equation with n-independent variables is

$$\sum_{i,j=1}^{n} A_{ij}u_{x_ix_j} + \sum_{i=1}^{n} B_iu_{x_i} + Fu = G, \qquad (**)$$

where $A_{ij} = A_{ji}, B_i, F$ and G are functions depending only on n-independent variables.

Homogeneous equation. if $G \equiv 0$.

Nonhomogeneous equation. if $G \not\equiv 0$.

The general solution of partial differential equations depending on arbitrary functions is different from ordinary differential equations depending on some constants.

Examples.

1° $u_{xy} = 0 \Rightarrow u_x(x,y) = f(x) \Rightarrow u(x,y) = g(x) + h(y)$, where $g(x)$ and $h(y)$ are arbitrary continuously differentiable functions.

2° Suppose $u = u(x,y,z)$ and $u_{yy} = 2$, then we can obtain the general solution

$$u(x,y,z) = y^2 + yf(x,z) + g(x,z),$$

where f and g are arbitrary continuously differentiable functions on two variables x, z.

Construction of the solution of n th-order partial differential equations is different from n th-order ordinary differential equations with finite (n) linearly independent functions, which can include infinite linearly independent functions.

Example. $u_x - u_y = 0$ using transformation on variables

$$\begin{cases} \xi = x + y, \\ \eta = x - y, \end{cases}$$

can get $2u_\eta = 0$ and obtain the general solution

$$u(x,y) = f(x+y),$$

where $f(x+y)$ is arbitrary continuously differentiable function which includes infinite functions, such as

$$(x+y)^n, \quad \sin n(x+y), \quad \cos n(x+y), \quad \exp n(x+y) \quad (n = 1, 2, 3, \cdots),$$

and these functions are linearly independent.

1.3 Linear operator

Operator. The mathematical operational rule by acting a function generates another function.

Example.

$$L[u] = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^3 u}{\partial y^3}, \qquad M[u] = \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2},$$

where $L = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^3}{\partial y^3}$ and $M = \frac{\partial^2}{\partial x^2} - x^2 \frac{\partial^2}{\partial y^2}$ are called differential operators.

$$P[u] = \int_a^b u(x,\tau) F(\tau, y) d\tau, \quad a \text{ and } b \text{ are constants,}$$

$$Q[u] = u(x,c) + u_x(x,c), \quad c \text{ is constant,}$$

where P is an integral operator and Q is a special operator which transforms a function with two variables x and y to another function with one variable x.

If operators A and B acting any functions of a set, can generate the same result, then A and B are called equivalent operators in this set and denoted by $A = B$.

Thus, we have

$$A[u] = B[u].$$

The definition of the **sum** of two differential operators A and B is that

$$(A + B)[u] = A[u] + B[u],$$

where u is a function.

The **product** of two operators A and B is the operator whose action to a function is the same with the action of B and A in sequence, namely

$$AB[u] = A(B[u]).$$

Differential operators satisfy the following four properties:

(1) commutative law of addition

$$A + B = B + A;$$

(2) associative law of addition

$$(A + B) + C = A + (B + C);$$

(3) associative law of multiplication

$$(AB)C = A(BC);$$

(4) distributive law of multiplication to addition

$$A(B + C) = AB + AC.$$

Except above results, in general, the following commutative law of multiplication

$$AB = BA$$

is not true. If all coefficients of differential operators are constants, the commutative of multiplication law is valid.

Example. Let

$$A = \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}, \qquad B = \frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y} \qquad (xy \not\equiv 0),$$

then

$$\begin{aligned} B[u] &= \frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}, \\ AB[u] &= \left(\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial x^2 \partial y^2} - y\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^3 u}{\partial y^3} - xy\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial y}. \end{aligned}$$

But

$$\begin{aligned} BA[u] &= \left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial y^2 \partial x^2} + x\frac{\partial^3 u}{\partial y^3} - y\frac{\partial^3 u}{\partial y \partial x^2} - xy\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

thus $AB[u] \neq BA[u]$.

Linear operators satisfy the following property:

$$L[au + bv] = aL[u] + bL[v],$$

Where a and b are constants.

The form of linear second order partial differential equation with two variables is

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y),$$

where A, B, C, D, E, F are the functions on variables x and y, $G(x,y)$ is the nonhomogeneous term.

If

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F,$$

then the equation can be written as follows:

$$L[u] = G, \quad \text{or} \quad Lu = G.$$

Exercises

1. For the following equations, specify that (a) which one is linear or quasilinear or nonlinear; (b) which one is homogeneous or nonhomogeneous; (c) the order of each equation.
   (a) $u_{xx} + xu_y = y$;
   (b) $uu_x - 2xyu_y = 0$;
   (c) $u_x^2 + uu_y = 1$;
   (d) $u_{xxxx} + 2x^2y^2u_{xxyy} + y^3u_{yyy} = 0$;
   (e) $u_{xx} + 4xyu_{xy} + u_{yy} = \sin x$;
   (f) $u_{xxx} + u_{xyy} + \ln u = 0$;
   (g) $u_{xx}^2 + u_y^2 + \sin u = e^y$.
2. Verify that two functions

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad u(x, y) = e^x \sin y$$

   are the solutions of the equation

$$u_{xx} + u_{yy} = 0.$$

3. Prove that $u = f(x, y)$ satisfies the equation

$$xu_x - yu_y = 0,$$

   where f is an arbitrary differential function, and verify that $\sin(xy)$, $\cos(xy)$, $\ln(xy)$, e^{xy} and $(xy)^3$ are the solutions of this equation.
4. Prove that $u = f(x)g(y)$ satisfies the equation

$$uu_{xy} - u_xu_y = 0,$$

   where f and g are arbitrary twice continuously differential functions.
5. Let $u_x = v$, find the general solution of the equation

$$u_{xy} + u_x = 0.$$

6. Let $u(x, y) = f(\lambda x + y)$ is the form of the solution of the following equation

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0,$$

   where λ is an unknown parameter. Find the general solution of this equation.

7*. Try to find the general solution of the equation

$$u_{yy} + u = 0.$$