MATHEMATICAL ANALYSIS (I)

数学分析 (I)

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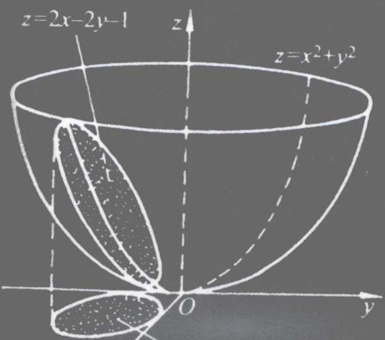

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内 容 提 要

本书是为贯彻教育部教学改革精神，实施全英语授课需要而编写，此书书稿在实际教学中已获得广泛好评。其内容包括：实数系统和函数；序列极限；函数极限及连续性；导数和微分；中值定理和导数的应用；不定积分；定积分；定积分的应用；微分方程初步。

本书可作为大学数学系及要求较高的专业的本科生教材，也可作为大学教师教学用书或教学参考书。

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Preface

To cultivate the science and technique personnel of high quality for the 21th century, great efforts have been made to the teaching reform in undergraduate courses and graduate courses in such aspects as teaching contents, teaching methods, teaching means and the course features, etc. , This is of great immediate significance to carrying quality-education forward to the full. As one of the main reform orientations, English teaching or bilingual teaching in mathematics courses such as mathematical analysis has shown its promising prospect. The enforcement of teaching material construction is a current impending mission in implementation along this line. The primary textbooks of mathematical analysis in western countries are somewhat different in style, structure and layout as compared with that taught in our universities. The former tend to develop the analysis theory in the setting of general metric space as well as in Euclidean space. The primary goal of writing this book is to match the content with the level accessible to undergraduate students in China.

Mathematical analysis is a fundamental subject facing all branches of science which needs mathematics. It has its beginnings in the rigorous formulation of calculus. Though the preliminaries of mathematical analysis may date back hundred years ago, it remains a classic study and a thorough treatment of the fundamentals of calculus. As foundation of modern mathematics, mathematical analysis is endowed with features of rigorous logicality and precise description.

Mathematical analysis is the branch of mathematics most explicitly concerned with the notion of a limit, either the limit of a sequence or the limit of a function. This subject is usually studied in the context of real numbers. However, it can also be defined and studied in any space of mathematical objects that is equipped with a definition of "closeness"-a topological space, or more specifically "distance"-a metric space.

This book is intended to display the structure of analysis as a subject in its own right. The main objective of the text is to introduce students to fundamental concepts and standard theorems of analysis and to develop analytical techniques for attacking problems that arise in mathematical theory and applications of mathematics. Due to restriction of academic level and lack of experience, there may be mistakes and neglects in this book. All comments and suggestions are heartily welcome.

The publication of this book benefited from the financial support of Shanghai Jiao Tong University Office of Academic Affairs, which I appreciate greatly. It is also pleasure to record thanks to Professor Han Zhengzhi, Editors Chen Kejian and Dai Baicheng of Shanghai Jiao Tong University Press for their valuable comments and suggestions. Special thanks are due Editor Sun Qikun who carefully read the entire manuscript and made technical modifications which led to an improved layout of this book.

Contents

Chapter 1 Real number system and functions

Function is the most fundamental research object of mathematical analysis, and functions and other concepts studied in our subject are based on real numbers in some way, so we begin our study of analysis with a discussion of the real number system and functions.

§ 1.1 Real number system

Most applications of mathematics use real numbers. For purposes of such applications, it suffices to think of a real number as a decimal. A *rational* number is one that may be written as a finite or infinite repeating decimal, such as

$$2, -\frac{7}{4} = -1.75, 2.2689, \frac{20}{3} = 6.666\ldots.$$

An *irrational* number has an infinite decimal representation whose digits form no repeating pattern, such as

$$\sqrt{3} = 1.732\,050\,808\ldots, \pi = 3.141\,592\,653\,5\ldots.$$

The rational numbers and irrational numbers together constitutes the *real numbers* (*real number system*).

We have four infinite sets of familiar objects, in increasing order of complication:

$\mathbb{N}$: *the natural numbers* are defined as the set $\{1, 2, \ldots, n, \ldots\}$.

$\mathbb{Z}$: *the integers* are defined as the set $\{0, \pm 1, \pm 2, \ldots,$

$\pm n, \ldots \}$.

$\mathbb{Q}$: *the rational numbers* are defined as the set $\{p/q: p, q \in \mathbb{Z}, q \neq 0\}$.

$\mathbb{R}$: *the set of real numbers* (or *the reals*) is composed of the rational numbers and the irrational numbers.

Remark (1) We have natural conclusions $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where each inclusion is proper;

(2) The irrational number set is $\mathbb{R} \setminus \mathbb{Q}$.

The real number line are often presented geometrically as points on a line (called the *real line* or the *real axis*). A point is selected to represent 0 and another to represent 1, as shown in Figure 1-1. This choice determines the scale. Under an appropriate set of axioms for Euclidean geometry, each point on the real line corresponds to one and only one real number and, conversely, each real number is represented by one and only one point on the line. It is customary to refer to the *point* x rather than the point representing the real number x.

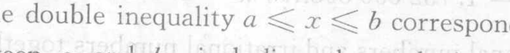

Figure 1-1

Geometrically, the inequality $x \leqslant b$ means that either x equals b or x lies to the left of b on the number line. The set of real numbers x that satisfy the double inequality $a \leqslant x \leqslant b$ corresponds to the line segment between a and b, including the endpoints. This set is sometimes denoted by $[a, b]$ and is called the closed interval from a to b. If a and b are removed from the set, the set is written as (a, b) and is called the open interval from a to b. The notation $(a, b]$ and $[a, b)$ etc. should be understood in a similar way.

Theorem 1.1.1 Given real number a and b such that $a \leqslant b + \varepsilon$ for every $\varepsilon > 0$. Then $a \leqslant b$.

Proof If $b < a$, take $\varepsilon = (a-b)/2$. Then

$$b+\varepsilon = b+\frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a,$$

which yields a contradiction. □

Definition Let $x_0 \in \mathbb{R}$. If $x_0 \in (a, b)$, then (a, b) is called a *neighborhood* of x_0, denoted by $U(x_0)$, and $(a, b) \setminus \{x_0\}$ is called a *free-center neighborhood* of x_0, denoted by $U^o(x_0)$. In particular, if $\delta > 0$, then $(x_0-\delta, x_0+\delta)$ is called a δ-neighborhood of x_0, denoted by $U(x_0, \delta)$, and $(x_0-\delta, x_0+\delta) \setminus \{x_0\}$ is called a *free-center δ-neighborhood* of x_0, denoted by $U^o(x_0, \delta)$ (δ may be called the *radius of the neighborhood*), i. e.

$$U(x_0, \delta) = \{x \mid |x - x_0| < \delta\},$$
$$U^o(x_0, \delta) = \{x \mid 0 < |x - x_0| < \delta\}.$$

Properties of ℝ We summarize the following properties of ℝ that we work with.

Addition We can add and subtract real numbers exactly as we expect, and the usual rules of arithmetic hold-such results as $x+y = y+x$.

Multiplication In the same way, multiplication and division behave as we expect, and interact with addition and subtraction in the usual way. So we have rules such as $a(b+c) = ab+ac$. Note that we can divide by any number except 0. We make no attempt to make sense of $a/0$, even in the case when $a = 0$, so for us $0/0$ is meaningless. Formally these two properties say that ℝ constructs a field algebraically, although it is not essential at this stage to know the terminology.

Order As well as the algebraic properties, ℝ has an ordering on it, usually written as "$a > 0$" or "$\geqslant$". There are three parts to the property:

(1) **Trichotomy** For any $a \in \mathbb{R}$, exactly one of $a > 0$, $a = 0$ or $a < 0$ holds, where we write $a < 0$ instead of the formally correct $0 > a$; in words, we are simply saying that a number is either positive, negative or zero.

(2) **Addition** The order behaves as expected with respect to addition: if $a > 0$ and $b > 0$ then $a + b > 0$; i. e. the sum of positives is positive.

(3) **Multiplication** The order behaves as expected with respect to multiplication: if $a > 0$ and $b > 0$ then $ab > 0$; i. e. the product of positives is positive.

Now we extend the real number system by adjoining two "ideal points" $+\infty$ and $-\infty$.

The symbols $+\infty$ ("plus infinity") and $-\infty$ ("minus infinity") do not represent actual real numbers. Rather, they indicate that the corresponding line segment extends infinitely far to the right or left. The symbol ∞ ("infinity") usually designates either $+\infty$ or $-\infty$. An inequality that describes such an infinite interval may be written as $[a, +\infty)$, $(-\infty, a)$, etc.

Definition By the extended real number system $\mathbb{R}^*$ we shall mean the set of real numbers $\mathbb{R}$ with two symbols $+\infty$ and $-\infty$ which satisfy the following properties:

(1) If $x \in \mathbb{R}$, then we have $x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$, $x - (+\infty) = -\infty$, $x - (-\infty) = +\infty$, $\frac{x}{+\infty} = 0$, $\frac{x}{-\infty} = 0$.

(2) If $x > 0$, then we have $x(+\infty) = +\infty$, $x(-\infty) = -\infty$.

(3) If $x < 0$, then we have $x(+\infty) = -\infty$, $x(-\infty) = +\infty$.

(4) $(+\infty) + (+\infty) = (+\infty)(+\infty) = (-\infty)(-\infty) = +\infty$, $(-\infty) + (-\infty) = (+\infty)(-\infty) = (-\infty)(+\infty) = -\infty$.

(5) If $x \in \mathbb{R}$, then we have $-\infty < x < +\infty$.

Note (1) As defined above, we denote $\mathbb{R} = (-\infty, +\infty)$, the set of real numbers, and $\mathbb{R}^* = [-\infty, +\infty]$, the set of extended real numbers. The points in $\mathbb{R}$ are said to be *finite* to distinguish them from the infinite points $-\infty$ and $+\infty$.

(2) For some of the later work concerned with limits, it is also convenient to introduce the terminology: Every open interval $(a, +\infty)$ is called a *neighborhood* of $+\infty$; every open interval $(-\infty, a)$ is called a *neighborhood* of $-\infty$.

Note that we write $a \geqslant 0$ if either $a > 0$ or $a = 0$. More generally, we write $a > b$ whenever $a - b > 0$.

Completion The set $\mathbb{R}$ has an additional property, which in contrast is much more mysterious-it is complete. It is this property that distinguishes it from $\mathbb{Q}$. Its effect is that there are always "enough" numbers to do what we want. Thus there are enough to solve any algebraic equation, even those like $x^2 = 2$ which can't be solved in $\mathbb{Q}$. In fact there are (uncountably many) more-all the numbers like π, certainly not rational, but in fact not even an algebraic number, are also in $\mathbb{R}$.

Definition Let $S \subseteq \mathbb{R}$ $(S \neq \varnothing)$.

(1) If there exists $t \in \mathbb{R}$ such that $x \leqslant t$ for any $x \in S$, then S is said to be *bounded above* and t is called an *upper bound* of S.

(2) Let t be an upper bound of S. If $t \leqslant d$ for any upper bound d of S, then t is called *the least upper bound* of S, which is denoted as t=l.u.b. S.

For example, let $S = \left\{ -\frac{1}{n} \middle| n = 1, 2, \ldots \right\}$. Then for any $d \in [0, +\infty)$, d is an upper bound of S, and l. u. b. $S = 0 \notin S$. Let $S = (0, 1]$. Then for any $d \in [1, +\infty)$, d is an upper bounds of S, and l.u.b. $S = 1 \in S$.

The completeness axiom Let $S \subseteq \mathbb{R}$ $(S \neq \varnothing)$. If S is bounded above, then there exists the least upper bound of S.

Definition The least upper bound of a number set S is also called *the supremum of* S, denoted as $\sup S$.

By the definition of the supremum, it is easy to check the following:

Remark Let $S \subseteq \mathbb{R}$ $(S \neq \varnothing)$.

(1) If $\sup S$ exists, it is unique;

(2) The following two statements are equivalent:

ⓘ $t = \sup S$;

ⓘⓘ for any $x \in S$, $x \leqslant t$, and for any $a < t$ there exists $x \in S$ such that $x > a$.

Example Let $S = \left\{ \frac{n}{n+1} \middle| n = 1, 2, 3, \cdots \right\}$. Then $\sup S = 1$.

Proof Clearly, for any $x \in S$, $x \leqslant 1$.

Now, let $a < 1$. Take $x = \frac{n}{n+1}$, where $n = \left[\frac{a}{1-a} \right] + 1$. Then $x \in S$ with $x > a$. Thus $\sup S = 1$. □

Note If a number set S has no upper bound, denote $\sup S = +\infty$.

Definition Let $S \subseteq \mathbb{R}$ $(S \neq \varnothing)$.

(1) If there exists $b \in \mathbb{R}$ such that $x \geqslant b$ for any $x \in S$, then S is said to be *bounded below*, and b is called a *lower bound* of S.

(2) Let b be a lower bound of S. If for any lower bound d of S, $b \geqslant d$, then b is called *the greatest lower bound* of S, which is denoted as $b = \text{g.l.b. } S$.

For example, let $S = \left\{ \frac{1}{n} \middle| n = 1, 2, \ldots \right\}$. Then for any $d \in (-\infty, 0]$, d is a lower bound of S, and $\text{g.l.b. } S = 0 \notin S$. Let $S = [1, 2)$. Then for any $d \in (-\infty, 1]$, d is a lower bound of S, and $\text{g.l.b. } S = 1 \in S$.

Theorem 1.1.2 Let $S \subseteq \mathbb{R}$ $(S \neq \varnothing)$. If S is bounded below, then there exists the greatest lower bound of S.

Proof Let $T=\{-x \mid x\in S\}$. Then T is bounded above. By the completeness axiom, there exists the least upper bound of T. Let $\beta=$l. u. b. T. It is easy to check that $-\beta=$g.l.b. S. □

Definition The greatest lower bound of a number set S is also called *the infimum of* S, denoted as inf S.

By the definition of the infimum, it is easy to check the following:

Remark Let $S\subseteq\mathbb{R}\,(S\neq\varnothing)$.

(1) If inf S exists, it is unique;

(2) The following two statements are equivalent:

ⓘ $b=\inf S$;

ⓘⓘ for any $x\in S$, $x\geqslant b$, and for any $a>b$ there exists $x\in S$ such that $x<a$.

Note If a number set S has no lower bound, denote inf $S=-\infty$.

Definition Let $S\subseteq\mathbb{R}\,(S\neq\varnothing)$. If there exists $t\in\mathbb{R}$ such that $|x|\leqslant t$ for any $x\in S$, then S is said to be *bounded*, and t is called a *bound* of S; otherwise, S is said to be *unbounded*.

Clearly, S is bounded if and only if S is both bounded above and bounded below.

Definition Let $S\subseteq\mathbb{R}\,(S\neq\varnothing)$.

(1) If there exists $\alpha\in S$ such that $x\geqslant\alpha$ for any $x\in S$, then α is called the *minimum* of S, denoted as $\alpha=\min S$;

(2) If there exists $\beta\in S$ such that $x\leqslant\beta$ for any $x\in S$, then β is called the *maximum* of S, denoted as $\beta=\max S$.

By the definitions of the minimum and the maximum of a number set, it is routine to check the following:

Remark Let $S\subseteq\mathbb{R}\,(S\neq\varnothing)$.

(1) $\sup S=\min\{y \mid x\leqslant y, \forall x\in S\}$; $\inf S=\max\{y \mid x\geqslant y,$

$\forall x \in S\}$.

(2) $\min S \in S$ and $\max S \in S$, but $\inf S$ or $\sup S$ may be not an element of S.

(3) $\inf S \in S$ if and only if $\inf S = \min S$; $\sup S \in S$ if and only if $\sup S = \max S$.

Examples (1) Let $S = \left\{ \frac{n}{n+1} \middle| n = 1, 2, \ldots \right\}$. Then $\inf S = \frac{1}{2} \in S$, $\sup S = 1 \notin S$, $\min S = \frac{1}{2}$, no max S.

(2) Let $S = \left\{ (-1)^n + \frac{(-1)^{n+1}}{n} \middle| n = 1, 2, \ldots \right\}$. Then $\inf S = -1 \notin S$, $\sup S = 1 \notin S$, no min S, no max S.

Theorem 1.1.3 Let A, $B \subseteq \mathbb{R}$ $(A, B \neq \varnothing)$. Then

(1) $\sup (A \cup B) = \max\{\sup A, \sup B\}$;

(2) $\inf (A \cup B) = \min\{\inf A, \inf B\}$.

Proof (1) If A or B has no upper bound, then $A \cup B$ has no upper bound, and in this case, $\sup (A \cup B) = +\infty = \max\{\sup A, \sup B\}$.

Now, we assume that both of A and B have upper bounds. For any $x \in A$, $x \in A \cup B$, and so $x \leqslant \sup (A \cup B)$. Thus, $\sup (A \cup B)$ is an upper bound of A. By the definition of the supremum, we have $\sup A \leqslant \sup (A \cup B)$; similarly, we have $\sup B \leqslant \sup (A \cup B)$. Hence, $\max\{\sup A, \sup B\} \leqslant \sup (A \cup B)$. On the other hand, let $x \in A \cup B$. Then $x \in A$ or $x \in B$, and so $x \leqslant \sup A$ or $x \leqslant \sup B$, i. e. $x \leqslant \max\{\sup A, \sup B\}$. Thus $\max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. So, we see that $\sup (A \cup B) \leqslant \max\{\sup A, \sup B\}$;

(2) can be proved by an analogous argument. □

Example Let $A \subseteq B$ $(\subseteq \mathbb{R})$ $(A, B \neq \varnothing)$. Then $\inf B \leqslant \inf A \leqslant \sup A \leqslant \sup B$.

Proof Clearly, $\inf A \leqslant \sup A$. Note $B = A \cup B$. We have sup B=sup$(A\cup B)$=max{sup A, sup B}$\geqslant$sup A. Similarly, inf $B\leqslant$ inf A, and then the result follows.

Theorem 1.1.4 (Dedekind gap theorem) Let $S, T\subseteq\mathbb{R}(S, T\neq\varnothing)$ such that $x\leqslant y$ for any $x\in S$ and any $y\in T$. Then

(1) sup $S\leqslant$ inf T;

(2) Moreover, the following three assertions are equivalent:

ⓘ There exists uniquely $c\in\mathbb{R}$ such that $s\leqslant c\leqslant t$ for any $s\in S$ and any $t\in T$;

ⓘⓘ sup S=inf T;

ⓘⓘⓘ For any $\varepsilon>0$, there exist $x\in S$ and $y\in T$ such that $y-x<\varepsilon$.

Proof (1) By the condition, for any $y\in T$, y is an upper bound of S, and so sup $S\leqslant y$ for any $y\in T$. Thus, sup S is a lower bound of T, which implies sup $S\leqslant$ inf T.

(2) We first show ⓘ$\Leftrightarrow$ⓘⓘ.

ⓘ$\Rightarrow$ⓘⓘ: If sup $S\neq$inf T, by (1) we have sup $S<$ inf T, and so there exist $x, y\in\mathbb{R}$ such that sup $S<x<y<$inf T. Thus, there exist two distinct numbers x and y such that $s\leqslant x\leqslant t$ and $s\leqslant y\leqslant t$ for any $s\in S$ and any $t\in T$, which yields a contradiction.

ⓘⓘ$\Rightarrow$ⓘ: Let c := sup S=inf T. Clearly, $s\leqslant c\leqslant t$ for any $s\in S$ and any $t\in T$. We further show such number c is unique. Assume that there exists $d\in\mathbb{R}$ such that $s\leqslant d\leqslant t$ for any $s\in S$ and any $t\in T$. Then, sup $S\leqslant d\leqslant$ inf T, and so d=sup S=inf T, i.e. $d=c$.

Now, we show ⓘⓘ$\Rightarrow$ⓘⓘⓘ.

ⓘⓘ$\Rightarrow$ⓘⓘⓘ: Note that for any $\varepsilon>0$, sup $S-\frac{\varepsilon}{2}$ is not an upper bound of S and inf $T+\frac{\varepsilon}{2}$ is not a lower bound of T. Thus, there