DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

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PREFACE

The rapid development of contemporary technology requires ever more extensive mathematical preparation for engineers. This has resulted in a demand for a more complete exposition of the applications of the fundamental mathematical disciplines for engineers, technicians, and students in technological institutions.

The present book examines a number of physical and technical problems which involve second-order partial differential equations. Considerable attention is also given to the theory of such equations. In addition, the text includes several chapters and sections of a general nature (indicated by an asterisk). The material in these sections does not as yet have direct application; nonetheless, it is important for an understanding of contemporary scientific literature on mathematical physics.

Among the applications studied are the vibrations of strings, membranes, and shafts; electric oscillations in lines; the electrostatic problem; the basic gravimetric problem; the emission of electromagnetic waves and their distribution along wave guides and in horns; the emission and dispersion of sound; gravity waves on the surface of a liquid; heat flow in a solid body, and so forth. Solutions are given to both very simple and more complicated problems, making it possible for the reader to master the methods considered in the book and also the physics of the phenomena in question. In almost every chapter, there are problems whose basic purpose is to develop the reader's technical skill.

Approximate methods for solving problems in mathematical physics are not discussed, since their exposition would require a considerable increase in the size of the book. Also excluded are certain specialized problems (for example, those associated with the physics of atomic reactors) that have arisen only in the last few years.

The preparation of the book was carried out under the guidance and with the cooperation of Member-Correspondent of the Academy of Sciences of the USSR, Professor Nikolai Sergeevich Koshlyakov, whose untimely death occurred before publication of the book. A noted specialist in the field of analytic number theory and higher transcendental functions, Prof. Koshlyakov published a number of works in the field of mathematical physics. In the course of his career, which included more than 30 years of scientific and pedagogical activity, as well as 15 years of research in applied problems, Prof. Koshlyakov always devoted a great deal of attention to the mathematical education of engineers. An excellent lecturer and teacher, he enjoyed the constant respect and devotion of his listeners and students. His textbook Osnovnye Differentsial'nye Uravneniya Matematicheskoi Fiziki (Basic Differential Equations of Mathematical Physics), several chapters of which are used in the present book, has seen four editions (the latest in 1936).

The authors of the Introduction and Parts I and III are N. S. Koshlyakov and M. M. Smirnov; the authors of parts II and IV are N. S. Koshlyakov and E. B. Gliner.

We take this occasion to express our deep gratitude to I. M. Gel'fand, G. I. Zel'tser and G. P. Samosyuk for graciously reading the individual sections of the book, to G. Yu. Dzhanelidze and S. I. Amosov for making a thorough review of the manuscript, and especially to Scientific Instructor G. P. Akilov. All of these made a number of valuable comments leading to an improvement in the text and to the correction of a number of errors.

Leningrad, September 5, 1951

E. B. Gliner
M. M. Smirnov

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INTRODUCTION

1. *The fundamental differential equations of mathematical physics*

Many problems in mechanics and physics involve the *study* of second-order partial differential equations. The following are some examples: (1) The study of various types of waves – elastic, acoustic, and electromagnetic – and of other oscillational phenomena leads to the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{1}$$

where c is the velocity of propagation of the wave in the given medium. (2) The processes of heat flow in a homogeneous isotropic body and other diffusion phenomena are described by the *heat-flow equation:*

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{2}$$

(3) Study of a steady thermal state in a homogeneous isotropic body leads to *Poisson's equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -f(x, y, z). \tag{3}$$

In the absence of internal heat sources, eq. (3) becomes *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{4}$$

The potentials of a gravitational or of a constant electric field in which there are no masses or electric charges also satisfy Laplace's equation.

Equations (1) - (4) are often called the fundamental equations of mathematical physics. A detailed study of these equations makes possible the theoretical treatment of a large number of physical phenomena and the solution of many physical and technical problems.

Each of eqs. (1) - (4) has an infinite number of particular solutions. In solving a specific physical problem, it is necessary to choose from among these solutions the one that satisfies certain additional conditions imposed by the physical situation. Thus, *problems in mathematical physics reduce to finding solutions to partial differential equations that satisfy certain additional conditions*. The most common of these additional conditions are the so-called *boundary conditions* (conditions that must be satisfied at the boundary of the medium in question) and *initial conditions* (which must be satisfied at the particular instant of time at which consideration of a physical process begins).

Let us note one very important point. *A problem in mathematical phys-*

ics is considered to be stated correctly if the problem has exactly one stable solution satisfying all the conditions. By "stable", we mean that small changes in any of the given conditions of the problem must cause a correspondingly small change in the solution. The existence and uniqueness requirement means that among the given conditions there are none that are incompatible and that these conditions are sufficient to determine a unique solution. The stability requirement is necessary for the following reason. In the given conditions for a specific problem, especially if they are obtained from experiment, there is always some error, and it is necessary that a small error in the given conditions causes only a small inaccuracy in the solution. This requirement expresses the physical determinacy of the stated problem.

Determining whether a problem in mathematical physics is stated correctly is a very important and at the same time extremely difficult question in the theory of partial differential equations. We shall not make a complete study of this question in the present book.

The following three sections are devoted to the classification of second-order equations of the form

$$\sum_{i,j=1}^{n} A_{ij}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) = 0\,,$$

and, in the case of two independent variables, to their reduction to canonical form.

2. *The reduction of second-order equations to canonical form*

Let us examine a second-order equation with two independent variables that is linear with respect to the second-order derivatives:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0\,, \tag{5}$$

where A, B, and C are functions of x and y that have continuous derivatives up to the second order.

Let us replace x and y by the new independent variables ξ and η. Suppose that

$$\xi = \varphi_1(x,y)\,,\qquad \eta = \varphi_2(x,y) \tag{6}$$

are twice continuously differentiable functions and that the Jacobian

$$\frac{\partial(\varphi_1,\varphi_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial\varphi_1}{\partial x} & \dfrac{\partial\varphi_1}{\partial y} \\ \dfrac{\partial\varphi_2}{\partial x} & \dfrac{\partial\varphi_2}{\partial y} \end{vmatrix} \neq 0 \tag{7}$$

throughout the region in question.

The derivatives with respect to the old variables are expressed in terms of the derivatives with respect to the new variables according to the formulae

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y},$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}, \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2}, \end{aligned} \tag{8}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}.$$

When we substitute the values of the derivatives in (8) into eq. (5) we obtain

$$\overline{A}(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} + 2\overline{B}(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \overline{C}(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} + \overline{F}\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right) = 0, \tag{9}$$

where

$$\begin{aligned} \overline{A}(\xi,\eta) &= A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2, \\ \overline{B}(\xi,\eta) &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}, \\ \overline{C}(\xi,\eta) &= A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \end{aligned} \tag{10}$$

By a direct substitution, we can easily verify that

$$\overline{B}^2 - \overline{A}\overline{C} = (B^2 - AC)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2 \tag{11}$$

In the transformation (6), the two functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are at our disposal. Let us show that it is possible to choose them so that only one of the following conditions will be satisfied:

$$1)\ \overline{A} = 0,\ \overline{C} = 0, \qquad 2)\ \overline{A} = 0,\ \overline{B} = 0, \qquad 3)\ \overline{A} = \overline{C},\ \overline{B} = 0.$$

Then, obviously, the transformed equation (9) will take the simplest form.

Let us examine the first-order differential equation

$$A\left(\frac{\partial \varphi}{\partial x}\right)^2 + 2B\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y} + C\left(\frac{\partial \varphi}{\partial y}\right)^2 = 0. \tag{12}$$

We must examine separately the cases $B^2 - AC > 0$, $B^2 - AC < 0$, and $B^2 - AC = 0$ throughout the entire region. The case in which the expression $B^2 - AC$ changes sign in the region will be examined later.

CASE I: $B^2 - AC$ greater than zero. In this case, eq. (5) is said to be of the *hyperbolic* type. We may assume that either $A \neq 0$ or $C \neq 0$. We shall examine separately the case when $A = C = 0$. Without loss of generality, we may assume that $A \neq 0$ everywhere. Then, eq. (12) may be written in the form

$$\left\{A\,\frac{\partial\varphi}{\partial x} + (B + \sqrt{B^2 - AC})\,\frac{\partial\varphi}{\partial y}\right\}\left\{A\,\frac{\partial\varphi}{\partial x} + (B - \sqrt{B^2 - AC})\,\frac{\partial\varphi}{\partial y}\right\} = 0\ .$$

This equation can be separated into two equations:

$$A\,\frac{\partial\varphi}{\partial x} + (B + \sqrt{B^2 - AC})\,\frac{\partial\varphi}{\partial y} = 0\ , \tag{12a}$$

$$A\,\frac{\partial\varphi}{\partial x} + (B - \sqrt{B^2 - AC})\,\frac{\partial\varphi}{\partial y} = 0\ . \tag{12b}$$

Consequently, the solutions of each of eqs. (12a) and (12b) will be solutions of eq. (12).

To integrate eqs. (12a) and (12b), we set up corresponding systems of differential equations [1)]

$$\frac{dx}{A} = \frac{dy}{B + \sqrt{B^2 - AC}}\ , \qquad \frac{dx}{A} = \frac{dy}{B - \sqrt{B^2 - AC}}\ ,$$

or

$$\begin{aligned} A\,dy - (B + \sqrt{B^2 - AC})\,dx &= 0\ ,\\ A\,dy - (B - \sqrt{B^2 - AC})\,dx &= 0\ . \end{aligned} \tag{13}$$

We note that eqs. (13) may be written in the form of a single equation

$$A\,dy^2 - 2B\,dx\,dy + C\,dx^2 = 0\ . \tag{13a}$$

Suppose that

$$\varphi_1(x, y) = \text{constant}, \qquad \varphi_2(x, y) = \text{constant} \tag{14}$$

are solutions of eq. (13). Then, as we know, their left members will be solutions of eqs. (12a) and (12b) and hence of eq. (12).

The curves representing (14) are called the *characteristic curves* or simply the *characteristics* of eq. (5), and eq. (12) is called the *equation of the characteristics*.

For an equation of the hyperbolic type ($B^2 - AC > 0$), the solutions (14) will be real and distinct. Here, we have two distinct families of real characteristics.

In eq. (6), let us set

$$\xi = \varphi_1(x, y)\ , \qquad \eta = \varphi_2(x, y)\ ,$$

where $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are solutions of eq. (12). Then, on the basis of (10), $\bar{A} = \bar{C} = 0$ in eq. (9). The coefficient $\bar{B}$ is everywhere different from zero in the region in question – a consequence of (7) and (11). Dividing eq. (9) by the coefficient $2\bar{B} \neq 0$, we reduce it to the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = F_1\left(\xi, \eta, u, \frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}\right)\ . \tag{15}$$

This is the *canonical form of an equation of the hyperbolic type*.

If $A = C = 0$, eq. (5) is of the hyperbolic type and is already in canonical form.

If eq. (5) is linear with respect to the first-order derivatives and to the function u itself, the transformed equation will also be linear:

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} + a(\xi,\eta)\frac{\partial u}{\partial \xi} + b(\xi,\eta)\frac{\partial u}{\partial \eta} + c(\xi,\eta)\, u = f(\xi,\eta)\,. \tag{16}$$

Setting

$$\xi = \mu + \nu\,, \qquad \eta = \mu - \nu\,,$$

we reduce eq. (15) to the form

$$\frac{\partial^2 u}{\partial \mu^2} - \frac{\partial^2 u}{\partial \nu^2} = \Phi\left(\mu,\nu,u,\frac{\partial u}{\partial \nu},\frac{\partial u}{\partial \mu}\right).$$

This is the *second canonical form* of an equation of the hyperbolic type.

CASE II: Suppose that $B^2 - AC = 0$ throughout the region in question. In this case, eq. (5) is of the *parabolic* type. We shall assume that throughout the region the coefficients in eq. (5) do not vanish simultaneously. The condition that $B^2 - AC = 0$ implies that at every point of this region one of the coefficients A and C is different from zero. Without loss of generality, we may assume that A is everywhere different from zero. Then, eqs. (12a) and (12b) are identical and take the form

$$A\frac{\partial \varphi}{\partial x} + B\frac{\partial \varphi}{\partial y} = 0\,. \tag{17}$$

It is easy to see that every solution of eq. (17), where $B^2 - AC = 0$, also satisfies the equation

$$B\frac{\partial \varphi}{\partial x} + C\frac{\partial \varphi}{\partial y} = 0\,. \tag{18}$$

We note that for an equation of the parabolic type the solutions (14) coincide, and we have only one family of real characteristics $\varphi_1(x,y) = \text{const.}$

Let us set

$$\xi = \varphi_1(x,y)\,,$$

where $\varphi_1(x,y)$ is a solution of eq. (17). For $\varphi_2(x,y)$, let us take any function such that the Jacobian $\partial(\varphi_1,\varphi_2)/\partial(x,y) \neq 0$. Since A is different from zero and, consequently, $\partial\varphi_1/\partial y$ is different from zero, we may take $\varphi_2 = x$. Then, on the basis of (10), $\bar{A}$ is identically equal to zero in eq. (9) and the coefficient of $\partial^2 u/\partial\xi\,\partial\eta$ is of the following form:

$$\bar{B} = \left(A\frac{\partial \varphi_1}{\partial x} + B\frac{\partial \varphi_1}{\partial y}\right)\frac{\partial \varphi_2}{\partial x} + \left(B\frac{\partial \varphi_1}{\partial x} + C\frac{\partial \varphi_1}{\partial y}\right)\frac{\partial \varphi_2}{\partial y}\,.$$

According to (17) and (18), $\bar{B}$ is identically equal to zero in the region in question. The coefficient $\bar{C}$ in eq. (9) is transformed to the form

$$\bar{C} = \frac{1}{A}\left(A\frac{\partial \varphi_2}{\partial x} + B\frac{\partial \varphi_2}{\partial y}\right),$$

and hence $\bar{C} \neq 0$, because otherwise, on the basis of eq. (17), the Jacobian $\partial(\varphi_1,\varphi_2)/\partial(x,y)$ would vanish. Dividing eq. (9) by $\bar{C} \neq 0$, we reduce it to the form

$$\frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{19}$$

This is the *canonical form of an equation of the parabolic type*.

If eq. (5) is linear, eq. (19) will also be linear:

$$\frac{\partial^2 u}{\partial \eta^2} + a_1(\xi,\eta)\frac{\partial u}{\partial \xi} + b_1(\xi,\eta)\frac{\partial u}{\partial \eta} + c_1(\xi,\eta)\, u = f_1(\xi,\eta). \tag{20}$$

CASE III. Suppose that $B^2 - AC < 0$ throughout the region in question. Eq. (5) is then said to be of the *elliptic* type *. It is easy to see that in this case the solutions (14) will be complex-conjugate and we shall not have real characteristics.

Let us set

$$\xi + i\eta = \varphi_1(x,y), \quad \xi - i\eta = \varphi_2(x,y),$$

where φ_1 and φ_2 are complex-conjugate functions satisfying eq. (12).

Making the substitution $\varphi_1(x,y) = \xi + i\eta$ in eq. (12), we obtain

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 - A\left(\frac{\partial \eta}{\partial x}\right)^2 - 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} - C\left(\frac{\partial \eta}{\partial y}\right)^2$$
$$+ 2i\left\{A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right\} = 0.$$

Setting the real and imaginary parts of this identity equal to zero, we obtain

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2,$$

$$A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0.$$

Hence, it follows on the basis of (10) that

$$\overline{A} = \overline{C}, \quad \overline{B} = 0$$

and, after division by $\overline{A} \neq 0$, eq. (9) takes the form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \tag{21}$$

This is the *canonical form of an equation of the elliptic type*.

If eq. (5) is linear, eq. (21) will also be linear:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + a_2(\xi,\eta)\frac{\partial u}{\partial \xi} + b_2(\xi,\eta)\frac{\partial u}{\partial \eta} + c_2(\xi,\eta)u = f_2(\xi,\eta). \tag{22}$$

Example. Let us consider the equation

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0 \quad (x > 0,\ y > 0). \tag{23}$$

* In the reduction of equations of the elliptic type to canonical form, we shall confine ourselves to analytic coefficients A, B, and C. Thus, we shall be able to find the solution to eqs. (12a) and (12b) in the form of an analytic function.