Ordinary Differential Equations with Modern Applications

THIRD EDITION

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Preface

This book is intended for an introductory course in ordinary differential equations. Its prerequisite is elementary calculus.

Perusal of the table of contents and the list of applications shows that the book contains the theory, techniques, and applications covered in the traditional introductory courses in ordinary differential equations. A major feature of this text is the quantity and variety of applications of current interest in physical, biological, and social sciences. We have furnished a wealth of applications from such diverse fields as astronomy, bioengineering, biology, botany, chemistry, ecology, economics, electric circuits, finance, geometry, mechanics, medicine, meteorology, pharmacology, physics, psychology, seismology, sociology, and statistics.

Our experience gained in teaching ordinary differential equations on the elementary, intermediate, and graduate levels at the University of Rhode Island convinced us of the need for a book at the elementary level which emphasizes to the students the relevance of the various equations that they are exposed to in the course. That is to say that the various types of differential equations encountered occur in the course of scientific investigations of real-world phenomena.

The goal of this book, then, is to make elementary ordinary differential equations more useful, more meaningful, and more exciting to the student. To accomplish this, we strive to demonstrate that ordinary differential equations are very much "alive" in present-day applications. This approach has indeed had a satisfying effect in the courses we have taught recently.

During the preparation and class testing of this text we continuously kept in mind both the student and the teacher. We have tried to make the presentation direct, yet informal. Definitions and theorems are stated precisely and rigorously, but theory and rigor have been minimized in favor of comprehension of technique. The general approach is to use a large number of routine examples to illustrate the new concepts, definitions, methods of solution, and theorems. Thus, it is intended that the material will be easily accessible to the student. The presence of modern applications in addition to the traditional applications of geometry, physics, and chemistry should be refreshing to the teacher.

Numerous routine exercises in each section will help to test and strengthen the student's understanding of the new methods under discussion. There are over 1200 exercises in the text with answers to odd-numbered exercises provided. Some thought-provoking exercises from *The American Mathematical Monthly, Mathematics Magazine,* and The William Lowell Putnam Mathematics Competition are inserted in many sections, with references to the source. These should challenge the students and help to train them in searching the literature.

Review exercises appear at the end of every chapter. These exercises will help the student review material presented in the chapter. Some of the review exercises have been taken directly from physics and engineering textbooks, in order to further emphasize that differential equations are very much present in applications and that the student is quite apt to encounter them in areas other than mathematics.

Every type of differential equation studied and every method presented is illustrated by real-life applications which are incorporated in the same section (or chapter) with the specific equation or method. Thus, the students will see immediately the importance of each type of differential equation that they learn how to solve. We feel that these "modern" applications, even if the student only glances at some of them, will help to stimulate interest and enthusiasm toward the subject of differential equations specifically and mathematics in general. Every application is intended to illustrate the relevance of differential equations outside of their intrinsic value as mathematical topics.

Many of the applications are integrated into the main development of ideas, thus blending theory, technique, and application. Frequently, the mathematical model underlying the application is developed in great detail. It would be impossible in a text of this nature to have such development for every application cited. Therefore, some of the applications are presented in considerable detail and some with little or no detail, as indicated in the list of applications. In practically all cases, references are given for the source of the model. Additionally, a large number of applications appear in the exercises; these applications are also suitably referenced. Consequently, applications are widespread throughout the book, and although they vary in depth and difficulty, they should be diverse and interesting enough to whet the appetite of every reader. We suggest that the instructor present only a few of the applications, while the rest will indicate to the student the relevance of differential equations in real-life situations.

In this Third Edition we have added to Chapter 1 a section on an elementary numerical method—Euler's method—in accordance with an increasing desire on the part of instructors to show students, early in the course, how to solve differential equations numerically. Along the same lines we have added computer exercises to Chapter 7, "Numerical Solutions of Differential Equations," so students can use the new technology in solving differential equations numerically. In addition, we have added new applications and applications-oriented exercises, incorporating topics in the sciences that have been prominent in recent years. We have also added more routine exercises to sections of the text where students need more drill.

This book contains adequate material for a two-semester course in differential equations. For a one-semester course we usually cover the following sections (in this list B stands for a brief discussion of the section and D stands for a detailed presentation): 1.1(B), 1.1.1(B), 1.2(B), 1.3(D), 1.3.1(B), 1.4(D), 1.4.1(B), 1.7(B), 1.7.1(B), 1.8(D), 2.1(B), 2.1.1(B), 2.2(D), 2.3(B), 2.4(D), 2.5(D), 2.6(B), 2.7(D), 2.8(D), 2.10(D), 2.11(D), 2.12(D), 3.1(B), 3.1.1(B), 3.2(D), 3.3(B), 4.2(D), 4.3(D), 5.2(B), 5.3(D), 5.4(D), 5.5(B), 6.1(B), 6.2(D), 7.3(B). For a one-quarter course the following sections are suggested: 1.1(B), 1.1.1(B), 1.2(B), 1.3(D), 1.3.1(B), 1.4(D), 1.4.1(B), 1.7(B), 1.7.1(B), 1.8(D), 2.1(B), 2.1.1(B), 2.2(D), 2.3(B), 2.4(D), 2.5(D), 2.6(B), 2.10(D), 2.11(D), 3.1(B), 3.1.1(B), 3.2(D), 5.2(B), 5.3(D), 5.4(D).

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CHAPTER 1

Elementary Methods—First-Order Differential Equations

1.1 INTRODUCTION AND DEFINITIONS

Differential equations are equations that involve derivatives of some unknown function(s). Although such equations should probably be called "derivative equations," the term "differential equations" (*aequatio differentialis*) initiated by Leibniz in 1676 is universally used. For example,

$$y' + xy = 3 \tag{1}$$

$$y'' + 5y' + 6y = \cos x \tag{2}$$

$$y'' = (1 + y'^2)(x^2 + y^2) \tag{3}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{4}$$

are differential equations. In Eqs. (1)–(3) the unknown function is represented by y and is assumed to be a function of the single independent variable x, that is, $y = y(x)$. The argument x in $y(x)$ (and its derivatives) is usually suppressed for notational simplicity. The terms y' and y'' in Eqs. (1)–(3) are the first and second derivatives, respectively, of the function $y(x)$ with respect to x. In Eq. (4) the unknown function u is assumed to be a function of the two independent variables t and x, that is, $u = u(t, x)$, $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$ are the second partial derivatives of the function $u(t, x)$ with respect to t and x, respectively. Equation (4) involves partial derivatives and is a *partial differential equation.* Equations (1)–(3) involve ordinary derivatives and are *ordinary differential equations.*

In this book we are primarily interested in studying ordinary differential equations.

DEFINITION 1

An ordinary differential equation of order n is an equation that is, or can be put, in the form

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where $y, y', \ldots, y^{(n)}$ are all evaluated at x.

The independent variable x belongs to some interval I (I may be finite or infinite), the function F is given, and the function $y = y(x)$ is unknown. For the most part the functions F and y will be real valued. Thus, Eq. (1) is an ordinary differential equation of order 1 and Eqs. (2) and (3) are ordinary differential equations of order 2.

DEFINITION 2

A solution of the ordinary differential equation (5) *is a function* $y(x)$ *defined over a subinterval* $J \subset I$ *which satisfies Eq.* (5) *identically over the interval* J.

Clearly, any solution $y(x)$ of Eq. (5) should have the following properties:

1. y should have derivatives at least up to order n in the interval J.
2. For every x in J the point $(x,y(x),y'(x), \ldots, y^{(n-1)}(x))$ should lie in the domain of definition of the function F, that is, F should be defined at this point.
3. $y^{(n)}(x) = F(x,y(x),y'(x), \ldots, y^{(n-1)}(x))$ for every x in J.

As an illustration we note that the function $y(x) = e^x$ is a solution of the second-order ordinary differential equation $y'' - y = 0$. In fact.

$$y''(x) - y(x) = (e^x)'' - e^x = e^x - e^x = 0.$$

Clearly, e^x is a solution of $y'' - y = 0$ valid for all x in the interval $(-\infty, +\infty)$. As another example, the function $y(x) = \cos x$ is a solution of $y'' + y = 0$ over the interval $(-\infty, +\infty)$. Indeed,

$$y''(x) + y(x) = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

In each of the illustrations the solution is valid on the whole real line $(-\infty, +\infty)$. On the other hand, $y = \sqrt{x}$ is a solution of the first-order ordinary differential equation $y' = 1/2y$ valid only in the interval $(0, +\infty)$ and $y = \sqrt{x(1 - x)}$ is a solution of the ordinary differential equation $y' = (1 - 2x)/2y$ valid only in the interval $(0, 1)$.

As we have seen, $y = e^x$ is a solution of the ordinary differential equation $y'' - y = 0$. We further observe that $y = e^{-x}$ is also a solution and moreover $y = c_1e^x + c_2e^{-x}$ is a solution of this equation for arbitrary values of the constants c_1 and c_2. It will be shown in Chapter 2 that $y = c_1e^x + c_2e^{-x}$ is the "general solution" of the ordinary differential equation $y'' - y = 0$. By the general solution we mean a solution with the property that any solution of $y'' - y = 0$ can be obtained from the function $c_1e^x + c_2e^{-x}$ for some special values of the constants c_1 and c_2. Also, in Chapter 2 we will show that the general solution of the ordinary differential equation $y'' + y = 0$ is given by $y(x) = c_1 \cos x + c_2 \sin x$ for arbitrary values of the constants c_1 and c_2.

In this chapter we present elementary methods for finding the solutions of some first-order ordinary differential equations, that is, equations of the form

$$y' = F(x, y), \tag{6}$$

together with some interesting applications.

The differential of a function $y = y(x)$ is by definition given by $dy = y'dx$.

With this in mind, the differential equation (6) sometimes will be written in the differential form $dy = F(x,y)dx$ or in an algebraically equivalent form. For example, the differential equation

$$y' = \frac{3x^2}{x^3 + 1}(y + 1)$$

can be written in the form

$$dy = \left[\frac{3x^2}{x^3 + 1}(y + 1)\right]dx \quad \text{or} \quad y' - \frac{3x^2}{x^3 + 1}y = \frac{3x^2}{x^3 + 1}.$$

There are several types of first-order ordinary differential equations whose solutions can be found explicitly or implicitly by integrations. Of all tractable types of first-order ordinary differential equations, two deserve special attention: differential equations with *variables separable,* that is, equations that can be put into the form

$$y' = \frac{P(x)}{Q(y)} \quad \text{or} \quad P(x)dx = Q(y)dy,$$

and *linear equations,* that is, equations that can be put into the form

$$y' + a(x)y = b(x).$$

Both appear frequently in applications, and many other types of differential equations are reducible to one or the other of these types by means of a simple transformation.

MATHEMATICAL MODELS 1.1.1

Differential equations appear frequently in mathematical models that attempt to describe real-life situations. Many natural laws and hypotheses can be translated via mathematical language into equations involving derivatives. For example, derivatives appear in physics as velocities and accelerations, in geometry as slopes, in biology as rates of growth of populations, in psychology as rates of learning, in chemistry as reaction rates, in economics as rates of change of the cost of living, and in finance as rates of growth of investments.

It is the case with many mathematical models that in order to obtain a differential equation that describes a real-life problem, we usually assume that the actual situation is governed by very simple laws—which is to say that we often make idealistic assumptions. Once the model is constructed in the form of a differential equation, the next step is to solve the differential equation and utilize the solution to make predictions concerning the behavior of the real problem. In case these predictions are not in reasonable agreement with reality, the scientist must reconsider the assumptions that led to the model and attempt to construct a model closer to reality.

First-order ordinary differential equations are very useful in applications. Let the function $y = y(x)$ represent an unknown quantity that we want to study. We know from calculus that the first derivative $y' = dy/dx$ represents the rate of change of y per unit change in x. If this rate of change is known (say, by experience or by a physical law) to be equal to a function $F(x, y)$, then the quantity y satisfies the first-order ordinary differential equation $y' = F(x, y)$. We next give some specific illustrations.

Biology

■ It has long been observed that some large colonies of bacteria tend to grow at a rate proportional to the number of bacteria present. For such a colony, let $N = N(t)$ be the number of bacteria present at any time t. Then, if k is the constant of proportionality, the function $N = N(t)$ satisfies the first-order ordinary differential equation[1]

$$\dot{N} = kN. \tag{7}$$

This equation is called the *Malthusian law* of population growth. T. R. Malthus observed in 1798 that the population of Europe seemed to be doubling at regular time intervals, and so he concluded that the rate of population increase is proportional to the population present. In Eq. (7) $\dot{N}$ stands for dN/dt. (As is customary, derivatives with respect to x will be denoted by primes and derivatives with respect to t by dots.) In this instance it is the time that is the independent variable. Equation (7) is a separable differential equation and its solution $N(t) = N(0)e^{kt}$ is computed in Example 3 of Section 1.3. Here $N(0)$ is the number of bacteria present initially, that is, at time $t = 0$. The solution $N(t)$ can be represented graphically as in Figure 1.1.

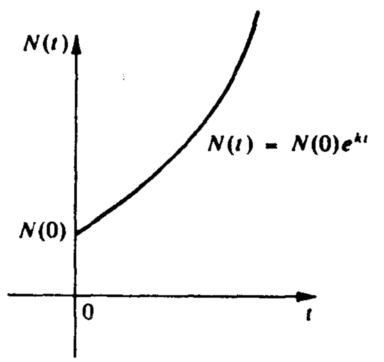

Figure 1.1

The following arguments may be used as a justification for Eq. (7). Let $N(t)$ be the total population at time t and let $N(t + \Delta t)$ be the total population at time $t + \Delta t$. Assume that the number of births and deaths during the small interval Δt are proportional to the product of the size of the population and the time interval Δt, that is,

$$\text{Birth} = bN(t)\Delta t \quad \text{and} \quad \text{Death} = dN(t)\Delta t,$$

[1]Since the function $N(t)$ takes on only integral values, it is not continuous and so not differentiable. However, if the number of bacteria is very large, we can assume that it can be approximated by a differentiable function $N(t)$, since the changes in the size of the population occur over short time intervals.

where b and d are positive constants. Then

$$N(t + \Delta t) - N(t) = (b - d)N(t)\Delta t$$

or

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = (b - d)N(t).$$

Taking limits as $\Delta t \to 0$ and setting $k = b - d$ lead to Eq. (7).

It should be emphasized that Eq. (7) is a mathematical model describing a colony of bacteria that grows according to a very simple, perhaps oversimplified, law: It grows at a rate proportional to the number of bacteria present at any time t. On the other hand, assuming this very simple law of growth leads us to a very simple differential equation. The solution, $N(t) = N(0)e^{kt}$, of Eq. (7) provides us with an approximation to the actual size of this colony of bacteria. Clearly, a more realistic mathematical model for the growth of this colony of bacteria is obtained if we take into account such realistic factors as overcrowding, limitations of food, and the like. Of course, the differential equation will then become more complex. It goes without saying that a mathematical model that is impossible to handle mathematically is useless, and consequently some simplifications and modifications of real-life laws are often necessary in order to derive a mathematically tractable model.

Pharmacology
Drug Dosages

■ It is well known in pharmacology[2] that penicillin and many other drugs administered to patients disappear from their bodies according to the following simple rule: If $y(t)$ is the amount of the drug in a human body at time t, then the rate of change $\dot{y}(t)$ of the drug is proportional to the amount present. That is, $y(t)$ satisfies the separable differential equation

$$\dot{y} = -ky, \tag{8}$$

where $k > 0$ is the constant of proportionality. The negative sign in (8) is due to the fact that $y(t)$ decreases as t increases, and hence the derivative of $y(t)$ with respect to t is negative. For each drug the constant k is known experimentally.

The solution of the differential equation (8) is (see Example 3 of Section 1.3)

$$y(t) = y_0e^{-kt}, \tag{9}$$

where $y_0 = y(0)$ is the initial amount (initial dose) of the drug. As we see from Eq. (9) (see also Figure 1.2), the amount of the drug in the patient's body tends to zero as $t \to \infty$. However, in many cases it is necessary to maintain (approximately) a constant concentration (and therefore approximately a constant amount) of the drug in the patient's body for a long time. To achieve this it is necessary to give the patient an initial booster dose y_0 of the drug and then at equal intervals of time, say every τ hours, give the patient a dose D of the drug.

[2]This model, as well as other mathematical models in medicine, is discussed by J. S. Rustagi in *Int. J. Math. Educ. Sci. Technol.* 2 (1971): 193–203.

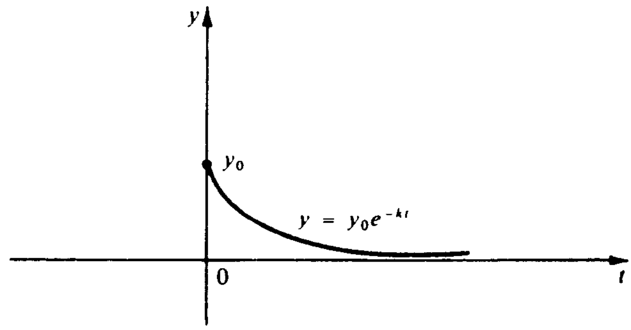

Figure 1.2

Equation (9) indicates the amount of the drug in the patient's body at any time t; hence, it is simple to determine the amount of the dose D. In fact, at time τ, and before we administer the dose D, the amount of the drug present in the body is

$$y(\tau) = y_0e^{-k\tau}.$$

If we want to maintain the initial amount y_0 of the drug in the body at the times τ, 2τ, 3τ, . . . , the dose D should satisfy the equation

$$y_0e^{-k\tau} + D = y_0.$$

Hence, the desired dose is given by the equation

$$D = y_0(1 - e^{-k\tau}). \quad (10)$$

Operations Research

■ Southwick and Zionts[3] developed an optimal control-theory approach to the education-investment decision which led them to the first-order linear (also separable) differential equation

$$\dot{x} = 1 - kx, \quad (11)$$

where x denotes the education of an individual at time t and the constant k is the rate at which education is being made obsolete or forgotten.

Psychology

■ In learning theory the separable first-order differential equation

$$\dot{p}(t) = a(t)G(p(t)) \quad (12)$$

is a basic model of the instructor/learner interaction.[4] Here G is known as the characteristic learning function and depends on the characteristics of the learner and of the material to be learned, $p(t)$ is the state of the learner at time t, and $a(t)$ is a measure of the intensity of instruction [the larger the value of $a(t)$ the greater the learning rate of the learner, but also, the greater the cost of the instruction].

[3] L. Southwick and S. Zionts, *Operations Res.* 22 (1974): 1156–1174.
[4] V. G. Chant, *J. Math. Psychol.* 11 (1974): 132–158.

Mechanics

■ Newton's second law of motion, which states that "the time rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the direction of this resultant force," implies immediately that the motion of any body is described by an ordinary differential equation. Recall that the momentum of a body is the product mv of its mass and its velocity v. If F is the resultant force acting on the body, then

$$\frac{d}{dt}(mv) = kF, \tag{13}$$

where k is a constant of proportionality. Equation (13) is an ordinary differential equation in v whose particular form depends on m and F. The mass m can be constant or a function of t. Also, F can be constant, a function of t, or even a function of t and v.

Electric Circuits

■ Kirchhoff's voltage law states that, "the algebraic sum of all voltage drops around an electric circuit is zero." This law applied to the RL-series circuit in Figure 1.3 gives rise to the first-order linear differential equation (see also Section 1.4)

$$L\dot{I} + RI = V(t), \tag{14}$$

where $I = I(t)$ is the current in the circuit at time t.

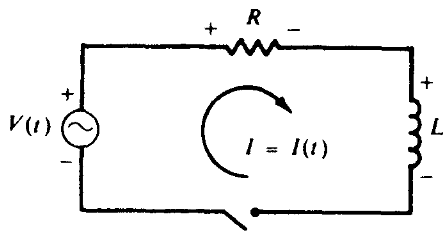

Figure 1.3

Orthogonal Trajectories

■ Consider the one-parameter family of curves given by the equation

$$F(x,y) = c. \tag{15}$$

Computing the differential of Eq. (15), we obtain

$$F_x dx + F_y dy = 0,$$

where F_x and F_y are the partial derivatives of F with respect to x and y, respectively. Thus,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{16}$$

gives the slope of each curve of the family (15). We want to compute another family of curves such that each member of the new family cuts each member

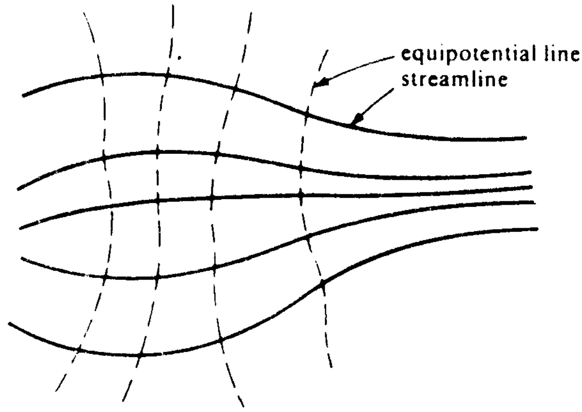

Figure 1.4

of the family (15) at right angles; that is, we want to compute the *orthogonal trajectories* of the family (15). In view of Eq. (16), the slope of the orthogonal trajectories of the family (15) is given by [the negative reciprocal of (16)]

$$\frac{dy}{dx} = \frac{F_y}{F_x}. \tag{17}$$

The general solution of Eq. (17) gives the orthogonal trajectories of the family (15).

There are many physical interpretations and uses of orthogonal trajectories:

1. In *electrostatic fields* the *lines of force* are orthogonal to the *lines of constant potential.*

2. In *two-dimensional flows of fluids* the lines of motion of the flow—called *streamlines*—are orthogonal to the *equipotential* lines of the flow (see Figure 1.4).

3. In *meterology* the orthogonal trajectories of the *isobars* (curves connecting all points that report the same barometric pressure) give the direction of the wind from high- to low-pressure areas.

Cell Dynamics ■ The red blood cells (erythrocytes) have a finite life span after which they are eliminated from circulation. Therefore, a constant supply of young erythrocytes produced by bone marrow is necessary. Let us denote by $R(t)$ the number of erythrocytes in the blood at time t and by $\dot{R}(t)$ its time derivative (rate of change). Then,

$$\dot{R}(t) = r^+(t) - r^-(t), \tag{18}$$

i.e., the rate of change is the rate of supply minus the rate of turnover (removal from circulation by organs like liver or spleen). It is further assumed that the turnover rate is proportional to the number of erythrocytes, $r^-(t) = bR(t)$, where b is a positive constant.