# THEORETICAL MECHANICS OF PARTICLES AND CONTINUA

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# PREFACE

This book developed from a first-year graduate course each of us has taught at Stanford since 1965. Most first-year physics graduate students enroll, along with some advanced undergraduates in physics and many graduate students from other departments. Originally, the course treated particle mechanics and mathematical physics, but the latter portion gradually evolved into a course on the physics of classical continuous media, not only for its own intrinsic interest but also as a natural outgrowth of the earlier material. We feel that a broad and thorough training in classical physics is essential for modern students of physics, independent of their subsequent choice of career. For example, familiarity with continuum mechanics, hydrodynamics, acoustics, and wave phenomena is fundamental in understanding the world around us, yet these subjects are generally missing from the standard graduate physics curriculum. In addition, classical mechanics provides a natural framework for introducing many of the advanced mathematical concepts in physics. A student's physical intuition concerning these everyday systems helps distinguish the mathematical questions from the physical ones, in contrast to the situation in classical electrodynamics or quantum mechanics, where the less familiar physics may itself be a source of difficulty.

We intend this frankly as a *textbook* and aim to provide a lucid and self-contained account of classical mechanics, together with appropriate mathematical methods. Although two quarters suffice to teach much of the material, a full year would allow a more complete and leisurely treatment. The material divides naturally into two parts: particles and continua. The first part starts from Newton's laws of motion and systematically develops the dynamics of classical particles, with chapters on basic principles, rotating coordinate systems, lagrangian formalism, small oscillations, dynamics of rigid bodies, and hamiltonian formalism, including a brief discussion of the transition to quantum mechanics. This part of the book also considers examples of the limiting behavior of many particles, facilitating the eventual transition to a continuous medium. The second part deals with classical continua, including chapters on strings, membranes, sound waves, surface waves on nonviscous fluids, heat conduction, viscous fluids, and elastic media. Each of these latter chapters is self-contained, providing the relevant physical background and developing the appropriate mathematical techniques. Thus the text treats lagrangian field theory, eigenfunctions and Sturm-Liouville theory, variational methods, perturbation theory, Green's functions, Fourier and Laplace transforms, and asymptotic techniques like the method of stationary phase. In addition, appendixes provide brief summaries of the theory of complex variables, vector and tensor calculus in curvilinear orthogonal coordinates, separation of variables, and integral representations of special functions.

Any treatment of classical mechanics must confront the question of special relativity. We have decided to omit it entirely, for we feel that it fits more naturally into classical electrodynamics, where the Lorentz invariance facilitates the treatment of four-dimensional space-time. In contrast, the customary relativistic generalization of Newton's laws of motion strikes us as cumbersome at best.

A textbook on mechanics faces a difficult problem in selecting references. Since our aim is to teach current physics for modern applications, we have not included primary sources, which students frequently find obscure or irrelevant. Some historical perspective is valuable, however, and we end this preface with a short chronological list of significant names associated with mechanics and mathematical physics. In addition, we have listed in Appendix G several familiar basic texts and monographs. These sources suffice for most sections, but where appropriate we have added selected references that we have found particularly clear or helpful, as a guide to further study.

Every chapter contains several homework problems of varying degrees of difficulty. We consider them an integral part of the text, and all students should attempt several from each chapter. Since classical mechanics is an old subject, no effort has been made to trace the origin of our examples and problems, many of which are modified versions of those in the list of texts and monographs.

The reader is assumed to be familiar with intermediate mechanics at the level of J. B. Marion, *Classical Dynamics of Particles and Systems*, 2d ed., Academic, New York, 1970, and with the elements of linear algebra and partial differential equations. A working knowledge of the first and second law of thermodynamics, at the level of F. Reif, *Fundamentals of Statistical and Thermal Physics*, McGraw-Hill, New York, 1965, will make some of the later sections on sound waves, heat conduction, and viscous fluids more meaningful.

We are grateful to our own teachers, in particular S. D. Drell and G. F. Carrier, for introducing us to many of these beautiful topics. We would also like to thank Victoria LaBrie for her invaluable help in the preparation of this manuscript.

*Alexander L. Fetter*
*John Dirk Walecka*

# SIGNIFICANT NAMES IN MECHANICS AND MATHEMATICAL PHYSICS†

Isaac Newton (1642–1727)
Daniel Bernoulli (1700–1782)
Leonhard Euler (1707–1783)
Jean Le Rond d'Alembert (1717–1783)
Joseph Louis Lagrange (1736–1813)
Pierre Simon de Laplace (1749–1827)
Adrien Marie Legendre (1752–1833)
Jean Baptiste Joseph Fourier (1768–1830)
Karl Friedrich Gauss (1777–1855)
Siméon-Denis Poisson (1781–1840)
Friedrich Wilhelm Bessel (1784–1846)
Augustin-Louis Cauchy (1789–1857)
George Green (1793–1841)
Carl Gustav Jacob Jacobi (1804–1851)
William Rowan Hamilton (1805–1865)
Joseph Liouville (1809–1882)
George Gabriel Stokes (1819–1903)
Hermann Ludwig Ferdinand Helmholtz (1821–1894)
Gustav Robert Kirchhoff (1824–1887)
William Thomson (Lord Kelvin) (1824–1907)
Georg Friedrich Bernhard Riemann (1826–1866)
John William Strutt (Lord Rayleigh) (1842–1919)

† Detailed accounts of their contributions can be found in C. C. Gillispie (ed.), "Dictionary of Scientific Biography," Scribners, New York, 1970.

# CONTENTS

CHAPTER
# ONE
# BASIC PRINCIPLES

Classical mechanics involves the application of Newton's laws of motion to explain and predict the dynamical motion of point particles and bulk continuous matter. As such, it concerns the behavior of familiar classical macroscopic objects—natural and artificial satellites, the atmosphere and the oceans, laboratory solids, and even the earth itself. Indeed, one principal aim in studying classical mechanics is to understand the everyday world and to learn how to describe its properties quantitatively. In addition, classical mechanics has proved basic in deriving quantum descriptions of atomic matter, far from the original realm of classical physics. Finally, the challenge of characterizing continuous media has stimulated much of the basic mathematics of modern theoretical physics. Thus the study of bulk systems provides a natural framework for introducing and illustrating these techniques.

## 1 NEWTON'S LAWS

Although Newton's laws of motion are easily stated, their full implications involve subtle and complicated nonlinear phenomena that remain only partially explored. Since these laws are central to all our subsequent work, we briefly review them and some of their most basic corollaries and consequences.

### Statement of Newton's laws

We first define a primary inertial coordinate system that is at rest with respect to the fixed stars. Newton's first law then states:

*In this primary inertial frame, every body remains at rest or in uniform motion unless acted on by a force* **F**. *The condition* $\mathbf{F} = 0$ *thus implies a constant velocity* **v** *and a constant momentum* $\mathbf{p} = m\mathbf{v}$.

In effect, Newton's first law asserts that such an inertial frame exists to arbitrary accuracy. If we construct an inertial frame and eliminate the forces as accurately as we can, Newton's first law appears to hold. Note that any experimental verification of this law must be approximate, for gravitational forces are always present in the universe as we know it.

Newton's second and third laws then state:

*In the primary inertial frame, application of a force alters the momentum, in an amount specified by the quantitative relation*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}} \tag{1.1}$$

Here a dot denotes a time derivative.

*To each action, there is an equal and opposite reaction. Thus if* $\mathbf{F}_{21}$ *is the force exerted on particle 1 by particle 2, then*

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{1.2}$$

*and these forces act along the line separating the particles.*

In applying these laws, several remarks are relevant. First, if mass is conserved and constant in time, the relation $\mathbf{p} = m\mathbf{v}$ reduces Eq. (1.1) to the familiar form

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{1.3}$$

Otherwise, it is essential to retain the original expression, e.g., in studying the dynamics of an evaporating droplet. Second, Eqs. (1.2) and (1.3) serve to define a given amount of mass in terms of a fundamental unit $m^*$ that acquires unit acceleration under the influence of a unit force. More precisely, if the standard particle 1 (mass $m^*$) interacts with any other particle ($m_2$, say), the magnitude of their relative accelerations $a_{12}$ and $a_{21}$ specifies $m_2$ through the relation $m_2|a_{12}| = m^*|a_{21}|$. These considerations are independent of the particular force law. Thus they apply to the gravitational force between two particles with masses $m_1$ and $m_2$

$$\mathbf{F}_{21} = -Gm_1m_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{1.4a}$$

with $G$ the universal constant of newtonian gravitation, and equally to Coulomb's force between two electrified objects with charges $Q_1$ and $Q_2$

$$\mathbf{F}_{21} = Q_1Q_2\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{1.4b}$$

in cgs units. It is striking that both these basic forces vary as the inverse square of the separation. It is the physicist's task to classify and enumerate the forces acting on a system; Newton's laws then allow one to calculate the subsequent motion.

As a final remark, we can verify the principle of galilean relativity that any frame moving with constant velocity relative to an inertial frame is again inertial. Thus, two observers moving uniformly with respect to each other and with respect to the primary inertial coordinate system infer the same basic laws of motion, at least in the usual case that $\mathbf{F}_{21}$ depends only on the vector separation of the particles.

PROOF Let $\mathbf{r}$ and $\mathbf{r}'$ be the coordinates as seen in two different frames moving with constant relative velocity $\mathbf{V}$. Evidently $\mathbf{r}' = \mathbf{r} + \mathbf{V}t$, so that $\mathbf{r}_i - \mathbf{r}_j = \mathbf{r}'_i - \mathbf{r}'_j$ and $\mathbf{F}_{ij} = \mathbf{F}'_{ij}$. Moreover, the usual rules of calculus ensure that $d^2\mathbf{r}/dt^2 = d^2\mathbf{r}'/dt^2$, implying that both the forces and the accelerations are the same in the two frames.

## Conservation Laws

It is possible to work directly with Newton's laws, but there are distinct conceptual advantages in introducing special derived quantities like linear and angular momentum and energy, which turn out to satisfy certain simple relations.

**Linear momentum** Equation (1.1) can be reinterpreted as the statement that the applied force determines the rate of change of $\mathbf{p}$. In particular, $\mathbf{p}$ is a constant vector whenever $\mathbf{F}$ vanishes, and this relation holds separately for each vector component.

**Angular momentum** Define the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.5a}$$

and assume that $m$ is constant, implying

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} \tag{1.5b}$$

The rate of change of $\mathbf{L}$ is given by

$$\dot{\mathbf{L}} = m\dot{\mathbf{r}} \times \mathbf{v} + m\mathbf{r} \times \dot{\mathbf{v}}$$

and the observation that $\dot{\mathbf{r}} = \mathbf{v}$ eliminates the first term on the right-hand side. Use of Eq. (1.3) then gives

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} \equiv \mathbf{\Gamma} \tag{1.6}$$

where $\mathbf{\Gamma}$ is the torque. Once again, we have obtained a vector conservation law, any specified component of angular momentum remaining constant whenever the corresponding component of torque vanishes. In contrast to $\mathbf{p}$, however, we note that $\mathbf{L}$ depends on the choice of coordinate frame, since a shift of the origin by $-\mathbf{r}_0$ transforms $\mathbf{r}$ into $\mathbf{r} + \mathbf{r}_0$ and $\mathbf{L}$ correspondingly becomes $m\mathbf{r} \times \mathbf{p} + m\mathbf{r}_0 \times \mathbf{p}$, where

$\mathbf{r}_0$ is assumed zero. The even more complicated case of transformation to moving coordinates will be considered in Chap. 2.

**Energy and work** Consider a static force field $\mathbf{F}(\mathbf{r})$ defined throughout some region of space. If a test particle is inserted at $\mathbf{r}$ and moved a small distance $d\mathbf{s}$, the work done on the particle is $dW = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{s}$. Consequently, the work in moving the test particle a finite distance from point 1 to point 2 along some particular path is just the line integral

$$W_{1\to 2} = \int_1^2 d\mathbf{s} \cdot \mathbf{F}(\mathbf{r}) \tag{1.7a}$$

In general, this relation cannot be simplified. For the special case that the particle starts at $\mathbf{r}_1$ and follows a dynamical trajectory that passes through $\mathbf{r}_2$, however, the element of length $d\mathbf{s}$ is then just $\mathbf{v}\,dt$, and the dynamical principle (1.3) allows us to integrate Eq. (1.7$a$) directly

$$W_{1\to 2} = \int_1^2 d\mathbf{s} \cdot \left(m\frac{d\mathbf{v}}{dt}\right) = \int_1^2 dt\, m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = m\int_1^2 dt\,\frac{d}{dt}\,\tfrac{1}{2}v^2 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \tag{1.7b}$$

independent of the intervening path. If $T \equiv \frac{1}{2}mv^2$ denotes the kinetic energy, the work done in moving a particle from 1 to 2 is precisely the increase in the kinetic energy $T_2 - T_1$.

This result can be sharpened if $\mathbf{F}(\mathbf{r})$ has the special form

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) \tag{1.8}$$

where $U$ is known as the *potential*. Such forces are called conservative; although they occur frequently, it is important to realize that they are quite restrictive, the scalar function $U(\mathbf{r})$ being specified by only one number at each point whereas a general vector field requires three. For such conservative forces, the right-hand side of Eq. (1.7$a$) is readily rewritten $-\int_1^2 d\mathbf{s} \cdot \nabla U(\mathbf{r})$, and the integrand is now just the differential change in $U$ in moving from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{s}$. Thus

$$-\int_1^2 d\mathbf{s} \cdot \nabla U(\mathbf{r}) = -\int_1^2 dU = -U_2 + U_1 \tag{1.9}$$

A combination with Eq. (1.7$b$) immediately yields the relation $T_2 - T_1 = -U_2 + U_1$, or, equivalently, the conservation law

$$T_1 + U_1 = T_2 + U_2 \tag{1.10}$$

for the total energy $E = T + U$ in the presence of conservative forces. To conclude this section, we may also recall two other equivalent criteria for conservative forces (see Prob. 1.1):†

$$\nabla \times \mathbf{F}(\mathbf{r}) = 0 \qquad \text{for all } \mathbf{r} \tag{1.11a}$$

$$\oint d\mathbf{s} \cdot \mathbf{F}(\mathbf{r}) = 0 \qquad \text{for all closed paths} \tag{1.11b}$$

† Problems will be found at the end of each chapter.