Operator Theory
Advances and Applications
249

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ISSN 0255-0156 ISSN 2296-4878 (electronic)
Operator Theory: Advances and Applications
ISBN 978-3-319-21017-9 ISBN 978-3-319-21018-6 (eBook)
DOI 10.1007/978-3-319-21018-6

Library of Congress Control Number: 2016940057

Mathematics Subject Classification (2010): 46E30, 47B34, 42B35, 42B20, 42B25

Printed on acid-free paper

This book is published under the trade name Birkhäuser.
The registered company is Springer International Publishing AG (www.birkhauser-science.com)

Birkhäuser

Operator Theory: Advances and Applications

Volume 249

Founded in 1979 by Israel Gohberg

More information about this series at http://www.springer.com/series/4850

Preface

This Volume 2 is a natural continuation of Volume 1. It contains variable exponent results for spaces less considered in the literature, in particular Hölder, Morrey–Campanato, and grand spaces. Some parts of Volume 2 may be read independently of Volume 1, but in general reading of Volume 2 requires notation and definitions given in Volume 1.

In this Volume 2 we cover the problem of the boundedness of maximal, singular, and potential operators in variable exponent Morrey and Herz spaces, including the case of unbounded underlying sets. We provide also a fine comparison of Morrey and Stummel spaces that is new for constant exponents.

Other new function spaces employed in this volume are weighted Iwaniec–Sbordone spaces and some new spaces that are based on close ideals, such as grand Morrey spaces and their generalizations. These spaces are well fit for the study of a wide range of problems of non-linear partial differential equations related to existence, uniqueness, and regularity. Among other results in the above-mentioned spaces presented here, it is worthwhile mentioning a complete characterization of weights governing the validity of Sobolev type theorem in weighted grand Lebesgue spaces defined, generally speaking, over spaces of homogeneous type (SHT), and the solution of trace problems for one and two-sided potentials with product kernels and strong fractional maximal functions. We emphasize that we give also weak and strong type estimates criteria for fractional and singular integrals (including similar integral transforms with product kernels). Fortunately, the initial definition of Iwaniec–Sbordone spaces on bounded sets allowed us to give the above-mentioned results in the form of criteria. In generalized grand Morrey spaces the boundedness of Hardy–Littlewood maximal operators, as well as of Calderón–Zygmund operators is established. In the above-mentioned spaces the boundedness of Riesz-type potential operators is obtained in the framework of homogeneous and non-homogeneous spaces. We explore also the boundedness of commutators of Calderón–Zygmund-type operators as well as commutators of fractional integrals with BMO functions in generalized grand Morrey spaces. These results are applied to establish the regularity of solutions to elliptic equations in non-divergence form with VMO coefficients by means of the theory of singular integrals and linear commutators.

All the above-mentioned results on grand Lebesgue spaces concern Iwaniec–Sbordone spaces in their original setting on bounded sets. In this volume the grand Lebesgue spaces on sets of infinite measure are also introduced and the boundedness of sublinear operators is established. At the same time, a new version of weighted grand Lebesgue space on bounded sets is introduced and statements similar to the above-mentioned results are derived.

One of the novelties of the present book is that we introduce new function spaces unifying variable exponent Lebesgue spaces and grand Lebesgue spaces. These spaces are non-reflexive, non-separable, and non-rearrangement invariant. The boundedness of maximal functions, Calderón–Zygmund singular integrals, and potentials in grand variable exponent Lebesgue spaces defined on SHT is obtained.

In Appendix we introduce the grand Bochner–Lebesgue spaces in the spirit of Iwaniec–Sbordone spaces and prove some of their properties.

The volumes are mainly written in the consecutive way of presentation of the material, but in some chapters, for reader's convenience, we recall definitions of some basic notions. Although we use unified symbols for notation in most of the cases, in some of the cases the notation in a chapter is specific for that chapter.

Acknowledgement

We wish to express our sincere gratitude to Dr. Tsira Tsanava from the A. Razmadze Mathematical Institute for her help in preparing some parts of the book in LaTeX.

The first and second named authors were partially supported by the Shota Rustaveli National Science Foundation Grant (Contract Numbers D/13-23 and 31/47).

The third named author was partially supported by the research project "Study of non-standard Banach spaces", ID-PPTA: 6326 in the Faculty of Sciences of the Pontificia Universidad Javeriana, Bogotá, Colombia.

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Basic Definitions and Notation from Volume 1

Definitions and Basic Properties of Variable Exponent Function Spaces

For an open set $\Omega \subseteq \mathbb{R}^n$, we let $L^{p(\cdot)}(\Omega, \varrho)$ denote the weighted space of measurable functions $f : \Omega \to \mathbb{C}$ with weight as a multiplier, i.e.,

$$\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{\varrho(x) f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\} < \infty. \tag{0.1}$$

The notation $L^{p(\cdot)}(\Omega)$ stands for $L^{p(\cdot)}(\Omega, 1)$.
We also use the notation $L^{p(\cdot)}_w(\Omega)$ for the spaces defined by the norm

$$\|f\|_{L^{p(\cdot)}_w(\Omega)} := \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{f(x)}{\lambda}\right|^{p(x)} w(x)\, dx \leqslant 1\right\}.$$

We often assume that either

$$1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty \quad \text{on} \quad \Omega, \tag{0.2}$$

or

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty \quad \text{on} \quad \Omega. \tag{0.3}$$

The following inequalities hold:

$$\|f\|^{p_+} \leqslant I_{p(\cdot)}(f) \leqslant \|f\|^{p_-}, \quad \text{if} \quad \|f\| \leqslant 1, \tag{0.4}$$

$$\|f\|^{p_-} \leqslant I_{p(\cdot)}(f) \leqslant \|f\|^{p_+}, \quad \text{if} \quad \|f\| \geqslant 1, \tag{0.5}$$

where the modular $I_{p(\cdot)}$ is given by

$$I_{p(\cdot)}(f) := \int\limits_{\Omega} |f(y)|^{p(y)} dy.$$

In the case of unbounded p we denote $\Omega_\infty = \{x \in \Omega : p(x) = +\infty\}$ and write

$$\|f\|_{p(\cdot)} = \|f\|_{(p)} + \|f\|_{L^\infty(\Omega_\infty)}, \tag{0.6}$$

where

$$\|f\|_{(p)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega\backslash\Omega_\infty} \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leqslant 1\right\}.$$

The local log-condition is

$$|p(x) - p(y)| \leqslant \frac{A}{\ln \frac{1}{|x-y|}} \quad \text{for all } x, y \in \Omega \text{ with } |x-y| \leqslant \frac{1}{2}. \tag{0.7}$$

For bounded sets Ω, it may be equivalently written as

$$|p(x) - p(y)| \leqslant \frac{A_1}{\ln \frac{D}{|x-y|}} \quad x, y \in \Omega, \quad D > \operatorname{diam} \Omega.$$

The condition

$$\left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right| \leqslant \frac{A_p}{\ln(e + |x|)}, \quad \text{for all } x \in \Omega. \tag{0.8}$$

for unbounded sets Ω is known as the decay condition.

Basic Notation

Everywhere in the sequel we use the following notation:

$\mathbb{N}$ is the set of all natural numbers;

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

$\mathbb{R}^n$ is the n-dimensional Euclidean space with the distance $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$;

$\mathbb{Z}$ is the set of all integers;

$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$;

$\overline{B}(x, r)$ is the closed ball with center x and radius r;

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$;

$e_{n+1} = (0, 0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$;

Ω is an open set in $\mathbb{R}^n$;

$\partial\Omega$ is the boundary of Ω;

$\mathcal{P}(\Omega)$ is the class of measurable functions $p : \Omega \to [1, \infty]$, non necessarily bounded;

$\mathbb{P}(\Omega)$ is the class of exponents $p \in \mathcal{P}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$;

$\mathcal{P}^{\log}(\Omega)$ is the set of bounded exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition;

$\mathbb{P}^{\log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p_- \leqslant p_+ < \infty$;

for unbounded sets Ω, by $\mathcal{P}_\infty(\Omega)$, $\mathbb{P}_\infty(\Omega)$, $\mathcal{P}^{\log}_\infty(\Omega)$, $\mathbb{P}^{\log}_\infty(\Omega)$, we denote the subsets of the corresponding sets of exponents introduced above which satisfy the decay condition;

in the case $\Omega = \mathbb{R}_+$ by $\mathcal{P}_{0,\infty}(\mathbb{R}_+)$ we denote the class of exponents $p \in \mathcal{P}(\mathbb{R}_+)$ satisfying the decay condition at the origin and infinity, as in (1.47);

$A_p(\mathbb{R}^n)$, $p = const$, is the usual Muckenhoupt class of weights, see (2.1);

$\mathbb{A}_{p(\cdot)}(\Omega)$ is the class of weights ϱ such that the maximal operator is bounded in the weighted spaces $L^{p(\cdot)}(\Omega, \varrho)$;

$\mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ is the class of weights ϱ satisfying the condition (2.3);

$\mathcal{A}_{p(\cdot)}(\Omega)$ is the class of restrictions to $\Omega \subset \mathbb{R}^n$ of weights $\varrho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$;

We usually write inf and sup instead of ess inf and ess sup, without danger of confusion;

The notation $A \approx B$ for $A \geqslant 0$ and $B \geqslant 0$ means the equivalence $c_1 A \leqslant B \leqslant c_2 A$ with positive c_1 and c_2 not depending on values of A and B.

Quasimetric Measure Spaces

(X, d, μ) always denotes a quasimetric space with a quasidistance d:

$$d(x,y) \leqslant c_{\rm t}[d(x,z) + d(z,y)] \tag{0.9}$$

and a Borel regular measure μ. In some chapters we admit a non-symmetric distance and then we use the constant $c_{\rm s} \geqslant 1$ from the condition

$$d(x,y) \leqslant c_{\rm s} d(y,x).$$

We denote $\ell = \operatorname{diam} X$. The following standard conditions are assumed to be fulfilled:

1) all the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ are measurable,
2) the space $C(X)$ of uniformly continuous functions on X is dense in $L^1(\mu)$.

In most of the statements we also assume that

3) the measure μ satisfies the doubling condition: $\mu B(x, 2r) \leqslant C \mu B(x,r)$. A measure satisfying this condition is called *doubling measure.* A quasimetric measure space with doubling measure is called *space of homogeneous type* (SHT) .

A measure μ on X is said to satisfy the *reverse doubling condition* (written $\mu \in RDC(X)$), if there exist constants $a > 1$ and $b > 1$ such that

$$\mu\big(B(x, ar)\big) \geqslant b\mu\big(B(x,r)\big) \tag{0.10}$$

for all x and r. An SHT (X, d, μ) is called an RD-space if μ satisfies the reverse doubling condition.

$\mathcal{D}(X)$ will stand for the set of functions in L^∞ on X with compact support.

By $E^{\complement}$ we denote the complement of a set E in X.

The conditions

$$\mu\left(B\left(x,r\right)\right) \leqslant c_1 r^n. \tag{0.11}$$

and

$$\mu B(x,r) \geqslant c_0\, r^N, \tag{0.12}$$

are known as the *upper and lower Ahlfors conditions*; the first one is also referred to as the *growth condition.* From the doubling condition it follows that

$$\frac{\mu B(x,\varrho)}{\mu B(y,r)} \leqslant C\left(\frac{\varrho}{r}\right)^N, \quad N = \log_2 C_\mu, \tag{0.13}$$

for all the balls $B(x,\varrho)$ and $B(y,r)$ with $0 < r \leqslant \varrho$ and $y \in B(x,r)$. From (0.13) it follows that the doubling condition implies the lower Ahlfors condition for any open bounded set $\Omega \subseteq X$ on which $\inf_{x\in\Omega} \mu B(x,\ell) > 0$, with $\ell = \operatorname{diam} \Omega$.

The Hardy–Littlewood maximal function is defined by

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\, d\mu(y).$$

The space $L^{p(\cdot)}(X) = L^{p(\cdot)}_\mu(X)$ on (X, d, μ) is defined in the standard way:

$$\|f\|_{L^{p(\cdot)}_\mu(X)} = \inf\left\{\lambda > 0 : \int\limits_X \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu(x) \leqslant 1\right\}. \tag{0.14}$$

In the quasimetric measure spaces setting we use two forms of the log-condition. By $\mathcal{P}^{\log}(X)$ we denote the set of μ-measurable functions which satisfy the standard *local log-condition* on (X, d, μ) :

$$|p(x) - p(y)| \leqslant \frac{C_p}{-\ln d(x,y)}, \quad d(x,y) \leqslant \frac{1}{2}, \quad x, y \in X, \tag{0.15}$$

and by $\mathcal{P}^{\log}_\mu(X)$ we denote the set of functions $p : X \to [1,\infty)$ which satisfy the condition

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} \tag{0.16}$$

for all $x, y \in X$ such that $\mu B(x, d(x,y)) < \frac{1}{2}$, but note that in different chapters different notation may be used for these conditions.

In the case $d(x,y) = d(y,x)$ the condition (0.16) is equivalent to its symmetrical form

$$|p(x) - p(y)| \leqslant \frac{A}{-\ln \mu B(x, d(x,y))} + \frac{A}{-\ln \mu B(y, d(x,y))}.$$

The log-condition in the form (0.16), which coincides with (0.15) in the Euclidean space, is more suitable in the context of general quasimetric measure spaces, because in some results it allows one to impose fewer restrictions on (X, d, μ).

The following embeddings hold:

$$\mathcal{P}^{\log}(X) \subseteq \mathcal{P}^{\log}_{\mu}(X), \tag{0.17}$$

or

$$\mathcal{P}^{\log}_{\mu}(X) \subseteq \mathcal{P}^{\log}(X), \tag{0.18}$$

according to whether the lower or upper Ahlfors condition holds (see Lemma 2.56 in Volume 1).

For $\Omega \subseteq X$ and $p \in \mathcal{P}^{\log}_{\mu}(\Omega)$ the estimate holds:

$$\left\|\chi_{B(x,r)}\right\|_{p(\cdot)} \leqslant C\ [\mu B(x,r)]^{\frac{1}{p(x)}} \tag{0.19}$$

for all $r \in [0, \operatorname{diam} \Omega]$ when Ω is bounded and for $r \in [0, a], a < \infty$; the estimate (0.19) is valid also for $p \in \mathcal{P}^{\log}_{\mu}$ if the lower Ahlfors condition holds. (See Lemma 2.57 in Volume 1.)

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