ELECTRICITY AND MAGNETISM

An Introduction to the Mathematical Theory

by

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ELECTRICITY AND MAGNETISM

PREFACE

This book has been written in response to suggestions from friends who have asked for a text-book on the subject adapted more particularly to the needs of candidates for Part I of the Mathematical Tripos.

A complete study of the theory of electricity and magnetism, as a logical mathematical development from experimental data, requires a knowledge of the methods of mathematical analysis far beyond what can reasonably be expected from most readers of an elementary text-book. The knowledge of pure mathematics assumed in the present volume amounts to little more than some elementary calculus and a few properties of vectors. The ground is restricted by this limitation. It covers the schedule for Part I of the Tripos, including the fundamental principles of electrostatics, Gauss's theorem, Laplace's equation, systems of conductors, homogeneous dielectrics and the theory of images; steady currents in wires; elementary theory of the magnetic field and the elementary facts about the magnetic fields of steady currents. There are also short chapters on induced magnetism and induction of currents.

From one standpoint it would be preferable that a book on a branch of Natural Philosophy should consist of a continuous logical development uninterrupted by 'examples'. But experience seems to indicate that mathematical principles are best understood by making attempts to apply them; and, as the purpose of this book is didactic, I have had no hesitation in interspersing examples through the chapters and giving the solutions of some of them. The text is based upon lectures given at intervals over a period of many years, and the examples are part of a collection which I began to make for the use of my pupils about forty years ago, drawn from Tripos and College Examination papers.

As regards notation, I felt much hesitation about abandoning the use of V for the potential of an electrostatic field; but the custom of using a Greek letter to denote the scalar potential of a vector field has become general, and the matter was decided for me when I found '$\boldsymbol{E} = -\textbf{\textit{grad}}\, \phi$' in the Cambridge syllabus.

I am greatly indebted to Mr E. Cunningham of St John's College for reading a large part of the text and making many appropriate criticisms and useful suggestions; and also to Dr S. Verblunsky of the University of Manchester for reading and correcting the proofs, and to the printers and readers of the University Press for careful composition and correction.

A. S. R.

Cambridge
November 1936

Table of Units

C.G.S. absolute unit of force = 1 dyne
C.G.S. absolute unit of work or energy = 1 erg

ELECTRICAL UNITS

Practical units		Equivalent absolute C.G.S. units	
		Electrostatic	Electro-magnetic
Charge	1 coulomb	3×10^9	10^{-1}
Potential or electromotive force	1 volt	$3^{-1} \times 10^{-2}$	10^8
Current	1 ampère	3×10^9	10^{-1}
Resistance	1 ohm	$3^{-2} \times 10^{-11}$	10^9
Capacity	1 farad	$3^2 \times 10^{11}$	10^{-9}
Inductance	1 henry	$3^{-2} \times 10^{-11}$	10^9
Rate of working	1 watt	10^7	10^7

One microfarad is one-millionth of a farad.

An electromotive force of 1 volt drives a current of 1 ampère through a resistance of 1 ohm and work is then being done at the rate of 1 watt or 10^7 ergs per second.

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Chapter I

PRELIMINARY MATHEMATICS

1·1. We propose in this chapter to give a brief account of some mathematical ideas with which the reader must be familiar in order to be able to understand what follows in this volume.

1·2. Surface and volume integrals. Though the process of evaluating surface and volume integrals in general involves double or triple integration and must be learnt from books on Analysis, yet in theoretical work in Applied Mathematics considerable use is made of surface and volume integrals without evaluation; and we propose here merely to explain what is implied when such symbols as

$$\int f(x, y, z)\, dS \quad \text{and} \quad \int f(x, y, z)\, dv$$

are used to denote integration over a surface and throughout a volume.

A definite integral of a function of one variable, say $\int_a^b f(x)\, dx$, may be defined thus: let the interval from a to b on the x-axis be divided into any number of sub-intervals $\delta_1, \delta_2, \dots \delta_n$, and let f_r denote the value of $f(x)$ at some point on δ_r; let the sum $\sum_{r=1}^{n} f_r \delta_r$ be formed and let the number n be increased without limit. Then, provided that the limit as $n \to \infty$ of $\sum_{r=1}^{n} f_r \delta_r$ exists and is independent of the method of division into sub-intervals and of the choice of the point on δ_r at which the value of $f(x)$ is taken, this limit is the definite integral of $f(x)$ from a to b.

In the same way we may define $\int f(x, y, z)\, dS$ over a given surface; let the given surface be divided into any number of small parts $\delta_1, \delta_2, \dots \delta_n$ and let f_r denote the value of $f(x, y, z)$

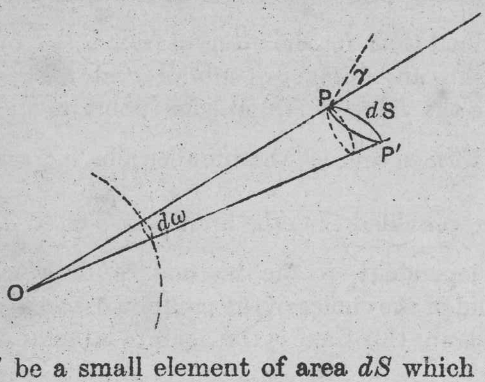
at some point on δ_r, then the limit as $n \to \infty$ of $\sum_{r=1}^{n} f_r \delta_r$, provided the limit exists under the same conditions as aforesaid, is defined to be the integral $\int f(x, y, z)\, dS$ over the given surface.

Any difficulty as to the precise meaning to be attributed to 'area of a curved surface' may be avoided thus: after choosing the point on each sub-division δ_r of the surface at which the value of $f(x, y, z)$ is taken, project this element of surface on to the tangent plane at the chosen point, and take the plane projection of the element as the measure of δ_r in forming the sum.

The integral $\int f(x, y, z)\, dv$ through a given volume may be defined in the same way.

1·3. Solid angles. The solid angle of a cone *is measured by the area intercepted by the cone on the surface of a sphere of unit radius having its centre at the vertex of the cone.*

The solid angle subtended at a point by a surface of any form is measured by the solid angle of the cone whose vertex is at the given point and whose base is the given surface.



Let PP' be a small element of area dS which subtends a solid angle $d\omega$ at O.

Let the normal to dS make an acute angle γ with OP and let $OP = r$. Then the cross-section at P of the cone which PP' subtends at O is $dS \cos\gamma$, and this cross-section and the small area $d\omega$ intercepted on the unit sphere are similar figures, so that

$$dS\cos\gamma : d\omega = r^2 : 1.$$

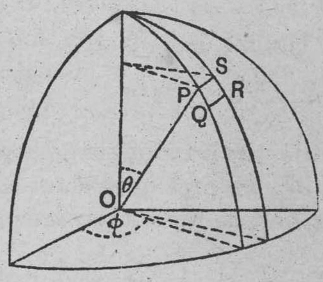
Whence
$$\left.\begin{aligned} d\omega &= (dS\cos\gamma)/r^2 \\ \text{or} \quad dS &= r^2\sec\gamma\, d\omega \end{aligned}\right\} \qquad \text{..................(1).}$$

It follows that the area of a finite surface can be represented as an integral over a spherical surface, thus

$$S = \int r^2 \sec\gamma\, d\omega \qquad \text{....................(2)}$$

with suitable limits of integration.

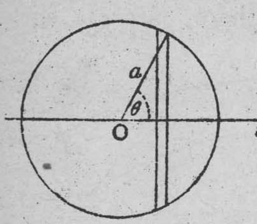
1·31. *dω* in polar co-ordinates. $d\omega$ is an element of the surface of a unit sphere. Let the element be $PQRS$ bounded by meridians and small circles, where the angular co-ordinates of P are θ, ϕ. Then since the arc PS subtends an angle $d\phi$ at the centre of a circle of radius $\sin\theta$, therefore $PS = \sin\theta\, d\phi$; and $PQ = d\theta$, so that

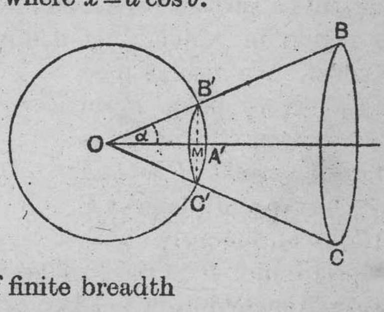


$$d\omega = PQ\,.\,PS = \sin\theta\, d\theta\, d\phi.$$

1·32. Solid angle of a right circular cone. A narrow zone of a sphere of radius a cut off between parallel planes may be regarded as a circular band of breadth $a\,d\theta$ and radius $a\sin\theta$, so that its area $= 2\pi a^2 \sin\theta\, d\theta$

$$= -2\pi a\, dx, \text{ where } x = a\cos\theta.$$





Hence the area of a zone of finite breadth

$= 2\pi a(x_1 - x_2)$

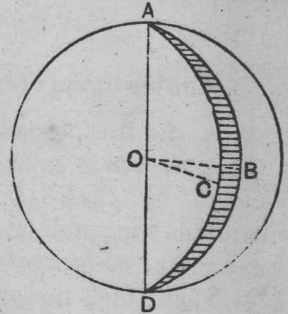
$=$ circumference of sphere $\times$ axial breadth of zone.

A right circular cone BOC of vertical angle 2α intercepts on a unit sphere of centre O a cap $B'A'C'$ of height $MA' = 1 - \cos\alpha$ and area

$$2\pi(1 - \cos\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

so that this is the measure of the solid angle of the cone.

1·33. The idea of the solid angle may easily be extended, if we observe that any bounded area on a unit sphere may be regarded as measuring a solid angle. Thus a lune bounded by semi-circles ABD, ACD may be taken as measuring the solid angle between the diametral planes ABD, ACD.



Let α be the angle between these planes. Because of the symmetry about AD, it is evident that

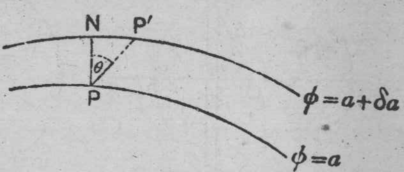
$$\text{area of lune} : \text{area of sphere} = \alpha : 2\pi.$$

But the area of the sphere is 4π, so that the area of the lune is 2α; or the solid angle between two planes is twice their inclination to one another.

1·4. Scalar functions of position and their gradients. Let $\phi(x, y, z)$ be a continuous single-valued function of the position of a point in some region of space. Suppose that the function ϕ is not constant throughout any region, so that the equation

$$\phi(x, y, z) = \text{const.}$$

represents a surface. We assume that through each point of the region in which ϕ is defined, there passes a surface $\phi = \text{const.}$ We also assume that at every point P on this surface there is a definite normal PN and that the tangent plane at P varies continuously with the position of P on the surface.



From the definition of ϕ two surfaces

$$\phi(x, y, z) = a \quad \text{and} \quad \phi(x, y, z) = b$$

cannot intersect; for if they had a common point it would be a point at which ϕ had more than one value, in contradiction to the hypothesis that ϕ is a single-valued function.

Consider two neighbouring surfaces

$$\phi = a \quad \text{and} \quad \phi = a + \delta a.$$

Let P, P' be points on each and let the normal at P to $\phi = a$ meet $\phi = a + \delta a$ in N. For small values of δa PN will also be normal to $\phi = a + \delta a$.

Then using ϕ_P to denote the value of ϕ at P, we have

$$\frac{\phi_{P'} - \phi_P}{PP'} = \frac{\delta a}{PP'} = \frac{\phi_N - \phi_P}{PP'} = \frac{\phi_N - \phi_P}{PN} \cdot \frac{PN}{PP'}$$

$$= \frac{\phi_N - \phi_P}{PN} \cos\theta,$$

where θ is the angle NPP'.

Now if $PP' = \delta s$ and $PN = \delta n$, and we make δa and therefore δs and δn tend to zero, the limit of $(\phi_{P'} - \phi_P)/PP'$ is the rate of increase of ϕ in the direction δs and is denoted by $\dfrac{\partial \phi}{\partial s}$; and similarly the limit of $(\phi_N - \phi_P)/PN$ is the rate of increase of ϕ in the normal direction δn and is denoted by $\dfrac{\partial \phi}{\partial n}$, and we have

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial n} \cos\theta \quad \text{.......................(1)}.$$

Thus we have proved that the space rate of increase of ϕ in any direction δs is the component in that direction of its space rate of increase in the direction normal to the surface $\phi = \text{const.}$; or that if we construct a vector of magnitude $\partial\phi/\partial n$ in direction PN, then the component of this vector in any direction is the space rate of increase of ϕ in that direction.

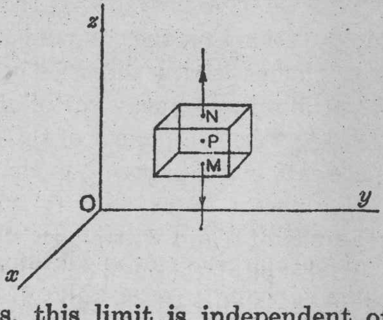
The vector $\partial\phi/\partial n$ with its proper direction is called the gradient of ϕ and written grad ϕ.

To recapitulate: ϕ is a continuous *scalar* function of position having a definite single value at each point of a certain region of space, and the *gradient of* ϕ is defined in this way: through any point P in the region there passes a surface $\phi = \text{const.}$, then *a vector* normal to this surface at P whose magnitude is the space rate of increase of ϕ in this normal direction is defined to be the *gradient of* ϕ at P, and it has the property that its component in any direction gives the space rate of increase of ϕ at P in that direction. It is clear that the gradient measures the greatest rate of increase of ϕ at a point.

1·5. A vector field. If to every point of a given region there corresponds a definite vector **A**, generally varying its magnitude and direction from point to point, then the region is called a **vector field**, or the field of the vector **A**; e.g. electric field, magnetic field.

1·51. Flux of a vector. If a surface S be drawn in the field of a vector **A** and A_n denotes the component of **A** normal to an element dS of the surface, then the integral $\int A_n dS$ is called the *flux of A through S*. Since a surface has two sides the sense of the normal must be taken into account, and the sign of the flux is changed when the sense of the normal is changed. The flux of a vector through a surface is clearly a scalar magnitude.

1·52. Divergence of a vector field. Let **A** denote a vector field which has no discontinuities throughout a given region of space. Let δv denote any small element of volume containing a point P in the region and let $\int A_n dS$ denote the outward flux of **A** through the boundary of δv, then the limit as $\delta v \to 0$ of $\dfrac{\int A_n dS}{\delta v}$ is defined to be the **divergence** of **A** at the point P and denoted by div **A**.



It can be shewn that, subject to certain conditions, this limit is independent of the shape of the element of volume δv, but for our present purpose, which is to obtain a Cartesian form for div **A**, it will suffice to calculate the limit for a rectangular element of volume. Using rectangular axes let P be the centre (x, y, z)

of a small rectangular parallelepiped with edges parallel to the axes of lengths δx, δy, δz.

Let the vector **A** have components A_x, A_y, A_z parallel to the axes at P.

Consider the contributions of the faces of the parallelepiped to the flux of the vector out of the element of volume. The two faces parallel to the xy plane are of area $\delta x\,\delta y$, the component of **A** normal to these at the centre (x, y, z) of the parallelepiped is A_z. The co-ordinates of the centres M, N of these faces are $x, y, z-\frac{1}{2}\delta z$ and $x, y, z+\frac{1}{2}\delta z$; so that if the magnitude of A_z at P is $f(x, y, z)$, its magnitude at M is $f(x, y, z-\frac{1}{2}\delta z)$ or $f(x, y, z)-\frac{1}{2}\frac{\partial f}{\partial z}\delta z$, to the first power of δz, i.e. $A_z-\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z$, and similarly the magnitude at N is $A_z+\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z$, and both these components are in the direction Oz. Then assuming what can easily be proved, that, subject to certain conditions, the average value of the component over each small rectangle is the value at its centre, the contributions of these two faces to the total *outward* flux are

$$-\left(A_z-\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z\right)\delta x\,\delta y \quad \text{and} \quad \left(A_z+\frac{1}{2}\frac{\partial A_z}{\partial z}\delta z\right)\delta x\,\delta y,$$

giving a sum

$$\frac{\partial A_z}{\partial z}\delta x\,\delta y\,\delta z.$$

Finding similarly the contributions of the other two pairs of faces, we have for the total outward flux

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\delta x\,\delta y\,\delta z$$

to this order of small quantities.

But the volume δv of the small element is $\delta x\,\delta y\,\delta z$, so that in accordance with our definition, dividing the flux by the volume and proceeding to the limit in which the terms of higher order in the numerator disappear, we have

$$\operatorname{div}\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \quad \text{...............(1).}$$