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Crystal Optics.

By

G. N. Ramachandran and S. Ramaseshan.

With 99 Figures.

A. Polarisation of light.

1. States of polarisation of light: Poincaré sphere. α) Light is a transverse electromagnetic wave and the nature of the vibration of the electric displacement vector in the plane normal to the direction of wave propagation defines the state of polarisation of a light beam. In a completely polarised beam[1], the vibration may be either linear in any azimuth at right angles to the propagation direction, or elliptical, with the major axis at any azimuth. The ratio of the axes of the ellipse can have any value and the sense of the ellipse may again be right or left handed. The two limiting cases of elliptic vibrations are linear and circular vibrations. Correspondingly, the light beam would be said to be elliptically, linearly or circularly polarised.

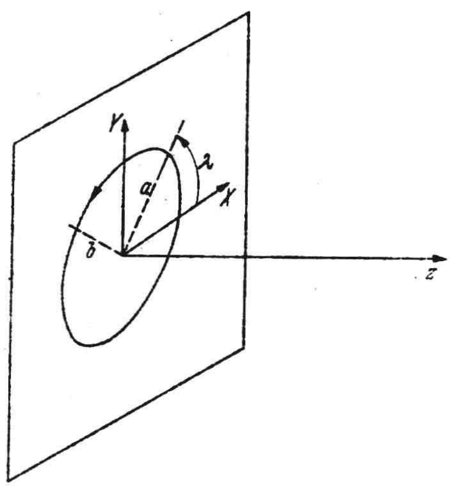


Fig. 1. Elliptically polarised light.

A general state of polarisation can thus be described by two quantities: (a) the orientation of the major axis of the ellipse, which may be specified by the angle λ which it makes with a given direction in the wave front and (b) the ratio of the axes of the ellipse (b/a, $b < a$). The sense of the ellipse could be specified by making the axial ratio positive for left-rotating ellipses and negative for right-rotating ellipses. The terms right and left-rotation are with respect to an observer looking towards the source of light. If the electric displacement vector rotates clockwise with progress of time, then it is right-rotating. At any instant of time the terminus of the electric displacement vector therefore forms a right-handed screw in space for a right elliptically polarised light beam.

Throughout this article, we shall imagine the light to be propagated along OZ (when not specified otherwise), which is taken to be horizontal (Fig. 1). The other two axes are taken horizontal (OX) and vertical (OY), the three together forming a right-handed system of co-ordinates.

The orientation of the major axis of the ellipse is given by the angle (λ) which it makes with the horizontal (OX) measured in the counter-clockwise direction,

[1] The descriptions of unpolarised and partially polarised beams of light are given in Sects. 8 and 11.

as seen by an observer looking towards the source. The ellipticity is defined by another angle ω, given by $\tan\omega = b/a$. The two angles λ and ω, which we shall denote by azimuth and ellipticity[1], uniquely specify the state of polarisation of a beam of light and all possible states of polarisation are covered by the range 0 to π of λ and the range $-\pi/4$ to $\pi/4$ of ω (taken together).

β) Poincaré sphere. The states of polarisation of a light beam can be uniquely represented by a point on the surface of a sphere of unit radius, whose latitude and longitude have the values 2ω, 2λ. This representation may be called the Poincaré representation and the sphere, the Poincaré sphere, after H. POINCARÉ who first suggested this idea[2]. The range of values of 2λ and 2ω required for describing all possible states of polarisation are therefore $2\lambda = 0$ to 2π, and $2\omega = -\pi/2$ to $\pi/2$, which covers the surface of the sphere completely. Thus all possible states of polarisation are represented by points on a sphere, there being a one-to-one correspondence between the points on the sphere and the various states of polarisation. A reversal of the direction of the major axis changes λ by π and therefore 2λ by 2π. It is the same state as before and is represented by the same point on the Poincaré sphere.

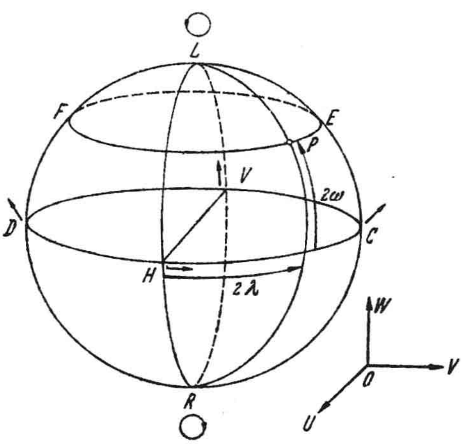


Fig. 2. The POINCARÉ sphere. A point P of longitude 2λ and latitude 2ω represents an elliptic vibration of azimuth λ and ellipticity ω.

Fig. 2 gives a picture of the Poincaré sphere. The points H and V represent horizontal and vertical linearly polarised light. Both are on the equator ($2\omega = 0$) and are at an angle π apart. L and R are the poles of the sphere and represent left and right circular vibrations. All linear states of polarisation are represented by points on the equator $HCVD$, the longitude being equal to twice the angle made with the horizontal. The points C and D, which are $\pi/2$ away from H and V thus correspond to linear vibrations at $\pm\pi/4$. All elliptical states having the same orientation (λ) of their major axes are represented by points on the meridian (LPR) of longitude 2λ. All ellipses having the same axial ratio ($b/a = \tan\omega$) are represented by points on the latitude circle (EPF) of latitude 2ω.

We shall, in general, call a beam of polarised light, whose state is represented by a point P on the Poincaré sphere, as light of polarisation state P. Similarly, a device which produces light of polarisation state P will be called "polariser P". A device which transmits light of polarisation state P completely is then called "analyser P". As will be seen later, it will be necessary to consider the orthogonal co-ordinate axes $OUVW$ in the space of the Poincaré sphere. These axes are respectively parallel to HV, DC and LR.

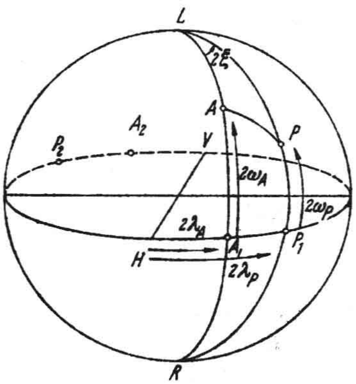
In crystal optics, we shall be interested in the changes produced in the state of polarisation of a beam of light traversing an anisotropic medium. The Poincaré representation is admirably suited for this purpose, and we shall therefore deal with some of the fundamental properties of the Poincaré sphere in this chapter.

[1] In spite of its ambiguity it has been decided to use the term "ellipticity" for the sake of convenience in preference to such terms as angle of ellipticity etc. When the "ellipticity" is small the ellipse is highly elongated, and it becomes a line in the limit when the "ellipticity" is zero.

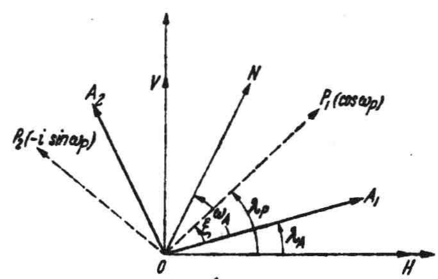
[2] H. POINCARÉ: Théorie Mathématique de la Lumière, Vol. II, Chap. XII. Paris 1892.

A knowledge of spherical trigonometry is required for this purpose, which may be readily obtained from the books listed in footnote[1]. Wherever possible, a perspective diagram of the sphere will be given, but for some purposes, the stereographic projection is more convenient. Details regarding the stereographic projection and its properties will be found in any textbook on crystallography, and the books listed in footnote[2] may be referred to in particular. The pole L is taken to be above the plane in all the projections; points on the sphere below the plane of the paper are indicated by a circle around the symbol representing the point, e.g. Ⓐ.

In spite of its elegance and simplicity, the Poincaré sphere representation of polarisation states is not discussed in most textbooks and works of reference on optics. An account of the Poincaré sphere and its use in the study of the transmission of light in optically active birefringent crystals is contained in POCKELS' Lehrbuch ([2], pp. 11–13 and 309–313). Since then, a fair number of original investigations appear to have made use of this representation[3]. The advantages of the Poincaré representation in studies on crystal optics and analysis of polarised light were pointed out in a recent paper of RAMACHANDRAN and RAMASESHAN[4]. A review of some of the application of the Poincaré sphere has been given by JERRARD, more recently[5].



a



b

Fig. 3 a and b. Light of state P is incident on an analyser A. Fraction of intensity transmitted is $\cos^2 \frac{1}{2} \widehat{PA}$.

2. Intensity transmitted by an analyser when light of arbitrary polarisation is incident on it[6]. In Fig. 3, let the analyser be represented by the state A, $(2\lambda_A, 2\omega_A)$. We wish to determine the fraction of a light beam of polarisation P, $(2\lambda_P, 2\omega_P)$

[1] W. J. MCLELLAND and T. PRESTON: A treatise on spherical trigonometry with applications to spherical geometry. London 1897. — I. TODHUNTER and J. G. LEATHEM: Spherical trigonometry. London 1911.

[2] S.L. PENFIELD: Amer. J. Sci. **11**, 1, 115 (1901); **14**, 249 (1902). — E. BOEKE: Die Anwendung der stereographischen Projection bei kristallographischen Untersuchungen. Berlin: Bornträger 1911. See also C. S. BARRETT: Structure of Metals. New York: McGraw-Hill 1943.

[3] J. BEQUEREL: Commun. Phys. Lab. Univ. Leiden No. 91C (1928); 221A (1930). — L. CHAUMONT: C. R. Acad. Sci., Paris **150**, 1604 (1913). — Ann. Chim. Phys. Paris (9) **4**, 101 (1915). — C.A. SKINNER: J. Opt. Soc. Amer. **10**, 490 (1925). — R.E. WRIGHT: J. Opt. Soc. Amer. **20**, 529 (1930). — G. BRUHAT and P. GRIVET: J. Phys. Radium **6**, 12 (1935). — Y. BJORNSTAHL: Phys. Z. **42**, 437 (1939). — Z. Instrumentenkde. **59**, 425 (1939). — O. SNELLMAN and Y. BJORNSTAHL: Kolloid-Beih. **52**, 403 (1941). — M.F. BOKOTEIN: J. Techn. Phys. USSR. **18**, 673 (1948). — G.N. RAMACHANDRAN and V. CHANDRASEKHARAN: Proc. Ind. Acad. Sci. A **33**, 199 (1951). — S. RAMASESHAN and V. CHANDRASEKHARAN: Current Sci. **20**, 150 (1951). — S. RAMASESHAN: Proc. Ind. Acad. Sci. A **34**, 32 (1951). — J. Ind. Inst. Sci. **37**, 195 (1955). — S. PANCHARATNAM: Proc. Ind. Acad. Sci., A **41**, 130, 137 (1955); A **42** 86, 235 (1955); A **44**, 247, 398 (1956); A **45**, 402; A **46**, 1, 280 (1957). — G. DESTRIAU and J. PROUTEAU: J. Phys. Radium **110**, 53 (1949).

[4] G.N. RAMACHANDRAN and S. RAMASESHAN: J. Opt. Soc. Amer. **42**, 49 (1952).

[5] H.G. JERRARD: J. Opt. Soc. Amer. **44**, 630 (1954).

[6] U. FANO: J. Opt. Soc. Amer. **39**, 859 (1949). — G.N. RAMACHANDRAN and S. RAMASESHAN: J. Opt. Soc. Amer. **42**, 49 (1952).

which is transmitted by this analyser. It is well known that a $\lambda/4$ plate with its slow axis OA_1 (Fig. 3b) at azimuth λ_A, followed by a linear analyser N at an angle ω_A to the slow axis, constitutes an elliptic analyser A. The action of the $\lambda/4$ plate is to reduce the ellipse A into a linear vibration parallel to the linear analyser and the ellipse A_a (antipodal to A) to a linear vibration perpendicular to it. When light of polarisation P is incident on this analyser it is easily seen that the light transmitted by it does not depend on the construction of the analyser, for an elliptic vibration P can be resolved into two orthogonal vibrations A and A_a in one and only one way, the intensity of the former component being transmitted by the analyser A. Hence without any loss of generality we may use the specific analyser described above for deducing the magnitude of the fraction transmitted.

This is done by resolving the incident light into two linear components P_1 and P_2 parallel to the axes of the ellipse, the latter lagging in phase by $\pi/2$. Thus the displacements along these two directions are for unit intensity

$$u_{P_1} = \cos\omega_P, \qquad u_{P_2} = -i\sin\omega_P. \tag{2.1}$$

The incident light resolved along OA_1 and OA_2 (the axes of the quarter wave plate) is therefore given by

$$\left.\begin{aligned} u_{A_1} &= \cos\omega_P\cos\xi + i\sin\omega_P\sin\xi, \\ u_{A_2} &= \cos\omega_P\sin\xi - i\sin\omega_P\cos\xi \end{aligned}\right\} \tag{2.2}$$

where (Fig. 3b)

$$\xi = (\lambda_P - \lambda_A).$$

On passage through the $\lambda/4$ plate a phase retardation $\pi/2$ is introduced between the vibrations along OA_1 and OA_2 and finally the linear analyser resolves the vibration into the plane ON giving the intensity transmitted by the analyser as

$$u_A = u_{A_1}\cos\omega_A + i\,u_{A_2}\sin\omega_A. \tag{2.3}$$

Thus the intensity transmitted by the analyser is

$$|u_A|^2 = \cos^2\xi\cos^2(\omega_A - \omega_P) + \sin^2\xi\sin^2(\omega_A + \omega_P).$$

This can be transformed, after some manipulation, into the form

$$|u_A|^2 = \tfrac{1}{2} + [\tfrac{1}{2}\sin 2\omega_P\sin 2\omega_A + \tfrac{1}{2}\cos 2\omega_P\cos 2\omega_A\cos 2(\lambda_P - \lambda_A)].$$

From the spherical triangle LPA of Fig. 3a we have the quantity within the square brackets to be equal to $\cos\widehat{PA}$, so that

$$|u_A|^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos\widehat{PA} \tag{2.4}$$

or

$$|u_A|^2 = \cos^2\tfrac{1}{2}\widehat{PA}. \tag{2.5}$$

Thus, the fraction of the intensity of light of the polarisation state P which is transmitted by the analyser A is $\cos^2\frac{1}{2}\widehat{PA}$ where $\widehat{PA}$ is the length of the arc joining P and A on the Poincaré sphere. This elegant result has a number of important applications, as will be seen below.

In particular, it is seen that if $\widehat{PA} = \pi$, i.e., the states of polarisation P and A are represented by opposite points on the Poincaré sphere, then no light is transmitted. Thus, these two states are orthogonal to one another. An analyser A transmits completely light of state A, while it completely cuts out light of state

A_a, A_a being the point antipodal to A. When arc $\widehat{PA}$ varies from 0 to π the transmitted fraction decreases from unity (P coincident with A) to zero (for P opposite to A). In particular, if A is a linear vibration, then the state A_a corresponds to the perpendicular linear vibration. If A is a left circular vibration corresponding to L, the orthogonal state is a right circular vibration, then A_a corresponds to R. If A corresponds to a general ellipse, then the orthogonal state A_a is the corresponding "crossed" ellipse which has its major and minor axes interchanged with respect to the former and has also the opposite sense of description.

In many applications, one is interested in the variations in the intensity of light transmitted by an analyser set close to extinction. In such a case, it is more convenient to consider the smaller arc $\widehat{PA_a}$ rather than the larger arc $\widehat{PA}$ which will be nearly π in value. The fraction of the intensity transmitted is then given by

$$t_a = \sin^2 \tfrac{1}{2} \widehat{PA_a}. \tag{2.6}$$

3. Effect of linear birefringence represented on the Poincaré sphere. In crystal optics a common problem that occurs is the following: When a beam of particular state of elliptic polarisation (P_1) is incident on a crystal plate, what will be the intensity and the state of polarisation P_2 of the emergent light. The crystal resolves the incident light into two specific polarised beams in different states of polarisation which are propagated with different velocities and, if the crystal is absorbing, with different absorption coefficients. In the case of a transparent crystal, the component beams will be in opposite states of polarisation A, A_a. When the specific states of opposite polarisation are linear, circular or elliptic, we shall refer to the medium as linearly, circularly or elliptically birefringent. One of the important results of the Poincaré representation, which makes it so useful in crystal optics, is that the state P_2 of the emergent light can be obtained from the state P_1 of the incident light by the simple geometrical operation of rotating the sphere about the axis $A A_a$ through an angle Δ, where Δ is the phase advance of A over A_a introduced by the crystal. We shall first consider the case of a linearly birefringent medium.

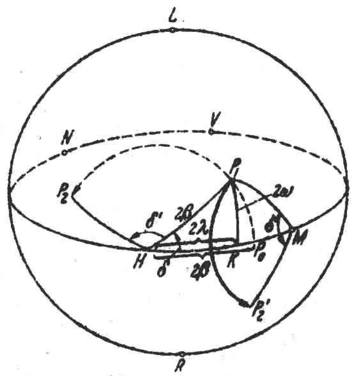


Fig. 4. Effect of linear birefringence. A phase difference δ' introduced between two linear orthogonal states M and N is equivalent, to an anti-clockwise rotation through an angle δ' about the faster state M.

Let the two linear states of polarisation which are propagated unchanged through the medium be H and V (Fig. 4) and let the phase difference introduced between them due to the passage through the medium be δ, H leading V by δ. Suppose unit intensity of linearly polarised light at azimuth β represented on the equator by P_0 ($HP_0 = 2\beta$ in Fig. 4) be incident on the crystal. This may be resolved along H and V giving the components $\cos\beta$ and $\sin\beta$. Let this be converted into an elliptical beam represented by the point P_1 as a result of the phase difference δ introduced. Let this ellipse have an azimuth λ and ellipticity ω. Resolving the vibration along H and V, we have, for unit intensity, the two amplitudes to be

$$u_1 = \cos\omega \cos\lambda + i \sin\omega \sin\lambda, \tag{3.1}$$

$$u_2 = \cos\omega \sin\lambda - i \sin\omega \cos\lambda, \tag{3.2}$$

while their phases ε_1 and ε_2 are given by

$$\tan\varepsilon_1 = \tan\omega\tan\lambda, \tag{3.3}$$

$$\tan\varepsilon_2 = -\tan\omega\cot\lambda. \tag{3.4}$$

The amplitudes of the two components must be equal to $\cos\beta$ and $\sin\beta$, so that we have

$$\left.\begin{aligned}\cos^2\beta &= \cos^2\omega\cos^2\lambda + \sin^2\omega\sin^2\lambda,\\ \sin^2\beta &= \cos^2\omega\sin^2\lambda + \sin^2\omega\cos^2\lambda.\end{aligned}\right\} \tag{3.5}$$

The two equations are equivalent and can be put in the form

$$\cos 2\beta = \cos 2\omega\cos 2\lambda. \tag{3.6}$$

The phase difference between the two is given by

$$\delta = \varepsilon_1 - \varepsilon_2,$$

so that

$$\tan(\varepsilon_1 - \varepsilon_2) = \frac{2\tan\omega}{1-\tan^2\omega}\,\frac{1}{2}(\tan\lambda + \cot\lambda) \tag{3.7}$$

and

$$\tan\delta = \tan 2\omega/\sin 2\lambda. \tag{3.8}$$

We thus have two relations (3.6) and (3.8) between the quantities ω, λ and β, δ. They can be interpreted very simply by saying that the point P_1 is obtained from P_0 by rotating it about the axis HV through an angle δ. Both Eqs. (3.6) and (3.8) can be verified to hold between the elements of the right angled spherical triangle HP_1K (Fig. 4).

Thus, starting from the linear polarisation state P_0, the effect of introducing a phase difference δ between the components H and V (H leading V by δ) is to rotate the representative point about the axis HV by an angle δ, measured anticlockwise looking from H to V. It follows from this that, if the initial state is represented by a point P_1, now considered as a general point, then the effect of a phase difference δ' between H and V is to bring P_1 to P_2 by a rotation through an angle δ' about HV.

So also, if the phase difference δ' is not between the linear states H and V but between the two states of linear polarisation of azimuth α and $\alpha+\pi/2$ represented on the Poincaré sphere by points M and N, of longitude 2α, $\pi+2\alpha$ on the equator, the representative point is rotated by an angle δ' about the axis MN (from P_1 to P_2').

Similarly, if a phase difference δ is introduced between left- and right-circular vibrations, the effect can readily be shown to be equivalent to rotating the sphere through an angle δ about LR. Suppose the incident beam is linearly polarised parallel to OX, represented by the point H on the equator. Following FRESNEL, we may resolve the linear vibration into two circular vibrations (which are in phase along OX). If the left rotating circle (L) is advanced in phase by $\delta/2$ while the other (R) is retarded by $\delta/2$ (phase difference $=\delta$), it may be shown that the two together will produce a linear vibration at azimuth $\delta/2$. The corresponding representative point remains on the equator, but is at longitude δ. It is obtained from the original state by a rotation through an angle δ about LR. The proof is directly generalised to any linear vibration. Considering any ellipse as made up of two linear vibrations at right angles but with a phase difference of $\pi/2$, it will be seen that both components will be rotated by $\delta/2$ by introducing a phase difference of δ between L and R. Thus the axial ratio of the ellipse is unaffected, but its azimuth is rotated[1] by $\delta/2$; the latitude of the representative point on the Poincaré sphere is unchanged but its longitude increases by δ. This is equivalent to rotating the point through an angle δ about LR.

Thus, the effect of linear or circular birefringence, and the consequent introduction of a phase difference δ between two orthogonal linear or circular states of polarisation, can be determined by finding the effect of a rotation of the Poincaré sphere through an angle δ about the appropriate axis of rotation. These results are in fact consequences of even more general properties regarding the addition of *any* two orthogonally polarised beams (see Sect. 4).

4. Coherent addition of polarised beams[2]. α) *Direct interference of two polarised beams*. Suppose we have a pair of orthogonal analysers A and A_a. Then it follows from the results (3.4) and (3.5) that the intensities transmitted by the two analysers would be constant for all states of polarisation (P) for which the arc $\widehat{PA}$ (and therefore also the arc $\widehat{PA_a}$) is the same. Thus, the locus of points on the Poincaré sphere representing the states of polarisation for which a definite fraction f is transmitted by the analyser A is a small circle of centre A and radius $\widehat{PA}$ where

$$\cos^2 \tfrac{1}{2} \widehat{PA} = f. \tag{4.1}$$

For all these states, the analyser A_a will transmit a fraction

$$\cos^2 \tfrac{1}{2} \widehat{PA_a} = \sin^2 \tfrac{1}{2} \widehat{PA} = 1 - f.$$

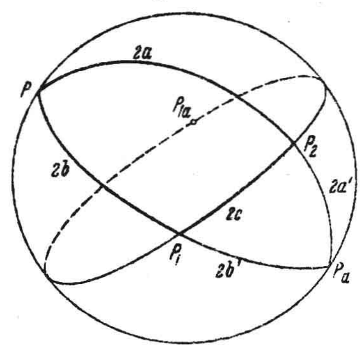


Fig. 5. Coherent addition of polarised beams. When a beam of intensity I and any state P is decomposed into two beams in the states P_1 and P_2, their intensities I_1 and I_2 are given by Eqs. (4.5) and (4.6); the phase difference is the supplement of half the area of the triangle $P_1 P_a P_2$.

The above result may be used to work out the resultant of the coherent addition of two beams of polarised light, say 1 and 2, whose states are represented by points P_1 and P_2 on the Poincaré sphere (Fig. 5) and whose intensities are I_1 and I_2 respectively. The resultant is the state P. Denote the arcs PP_2, PP_1, and P_1P_2 by $2a$, $2b$, $2c$ respectively, and similarly the arcs P_aP_2 and P_aP_1 by $2a'$, $2b'$ respectively. Let P_{1a} be the state opposite to P_1 and resolve the beam 2 into the state P_1 and P_{1a}, the intensities of which will be $I_2 \cos^2 c$ and $I_2 \sin^2 c$ respectively. The intensity of the resolved component of the combined beam along P may be obtained by the usual formula for combining two vibrations in the same state. The resultant intensity is

$$I_{P_1} = I_1 + I_2 \cos^2 c + 2\sqrt{I_1 I_2} \cos c \cos \delta. \tag{4.2}$$

The intensity of the resolved component of the combined beam in the state P_{1a} is

$$I_{P_{1a}} = I_2 \sin^2 c. \tag{4.3}$$

Since the beams of intensity I_{P_1} and $I_{P_{1a}}$ are orthogonal, the resultant intensity is just the sum of the two, independent of the phase difference between them. Thus,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos c \cos \delta \tag{4.4}$$

[1] This uses the fact that the phase difference *between* the components is unaltered by the operation of rotation. We shall not prove this, as a general proof for elliptic birefringence is given in Sect. 4.

[2] S. Pancharatnam: Proc. Ind. Acad. Sci. A **44**, 247 (1956).

and we may conveniently refer to δ as the phase difference between the two beams themselves though they are in different states of polarisation.

Now, the intensities of the resolved component of the resultant I in the state P_{1a} and of I_2 also in the state P_{1a} must be equal, since P_1 is orthogonal to P_{1a}. Hence

$$I \sin^2 b = I_2 \sin^2 c$$

or

$$I_2 = I \sin^2 b / \sin^2 c. \tag{4.5}$$

Similarly,

$$I_1 = I \sin^2 a / \sin^2 c. \tag{4.6}$$

Hence

$$\cos\delta = \frac{I - I_1 - I_2}{2\sqrt{I_1 I_2}\cos c} = \frac{\sin^2 c - \sin^2 b - \sin^2 a}{2\sin a \sin b \cos c} \tag{4.7}$$

$$= \frac{1 - \cos^2 c - \cos^2 b' - \cos^2 a'}{2\cos a' \cos b' \cos c} \tag{4.8}$$

or

$$\cos\delta = \cos\tfrac{1}{2}\varepsilon' \tag{4.9}$$

where ε' is the spherical excess or area of the spherical triangle $P_1 P_2 P_a$ which is colunar to the triangle $P P_1 P_2$. Thus

$$\delta = \pi \pm \tfrac{1}{2}\varepsilon'. \tag{4.10}$$

In particular, when $\delta = 0$, $\frac{1}{2}\varepsilon' = \pi$ or the spherical excess is 2π. The points P and P_a must then lie on the great circle passing through P_1 and P_2, P lying on the shorter arc $P_1 P_2$.

Thus, given I_1, I_2 and δ, one can first calculate I from Eq. (4.4) and then the spherical arcs a and b from Eqs. (4.5) and (4.6) which immediately fix the representative point P of the resultant, except for an ambiguity in the sign of δ, which is present also in Eq. (4.10). The ambiguity can be removed by a consideration of the combination of orthogonal states and a comparison with the conventions adopted in Sect. 3.

Suppose P_2 tends to the point P_{1a} i.e., $2c \to \pi$. Then, the triangle $P_1 P P_2$ becomes a lune in the limit (Fig. 6a). Denote the angle between the great circles $P_1 P_2 P_{1a}$ and $P_1 P P_{1a}$ at P as Δ. Then the spherical excess of the colunar triangle is $\varepsilon' = 2(\pi - \Delta)$. Thus, we have

$$\Delta = \pm\delta. \tag{4.11}$$

Further since the beams are orthogonal

$$I = I_1 + I_2 \tag{4.12}$$

and

$$\left.\begin{aligned} I_1/I &= \sin^2 b = \cos^2 a, \\ I_2/I &= \sin^2 a = \cos^2 b. \end{aligned}\right\} \tag{4.13}$$

If the phase relationship is kept constant and I_2/I_1 is altered, the resultant state moves along the locus for which Δ is constant i.e. along a great circle (e.g. $P_1 P P_{1a}$ of Fig. 6a). On the other hand, if the ratio I_2/I_1 is given and the phase difference δ is varied, then the resultant occurs in a small circle whose axis is $P_1 P_2$ (i.e. $P_1 P_{1a}$). It is however necessary to define the condition when the two have the same phase, which may be done by taking some great circle through $P_1 P_2$ as the standard of reference (say the one marked $\delta = 0$ in Fig. 6a). Then, for any given δ, the resultant P lies on a great circle rotated from the standard through an angle δ. Thus two position are possible corresponding to $\Delta = \pm\delta$.

We have already shown (Sect. 3) that for the case of linear birefringence, the upper positive sign is to be taken if P_1 leads P_2 in phase. From considerations of analytical continuity the same must be true for adjacent axes of rotation and hence for any axis of rotation of the Poincaré sphere. We have thus proved the proposition stated in Sect. 3 namely that *the effect of any elliptic birefringence is represented by an anticlockwise rotation about the point representing the faster state.*

This result for orthogonal vibrations may be used to resolve the ambiguity in (4.10) for the case of non-orthogonal vibrations by the method of analytical continuity, giving

$$\delta = \pi - \tfrac{1}{2}\varepsilon' \tag{4.14}$$

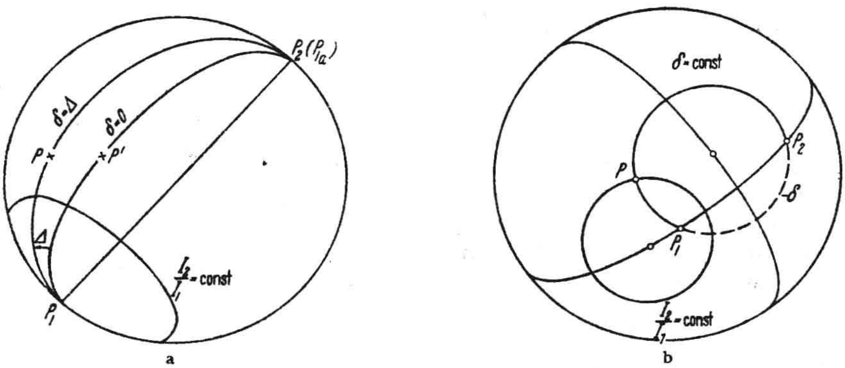


Fig. 6 a and b. Locus of the resultant state of polarisation P when the ratio of the intensities of two beams P_1 and P_2 or their phase difference is varied the other remaining constant. (a) States P_1, P_2 of the combining beams are orthogonal. (b) States P_1, P_2 non-orthogonal.

where ε' is to be counted positive if the sequence of points $P_1 P_a P_2$ (and therefore the sequence $P_1 P_2 P$) is described in a counter-clockwise sense on the surface of the sphere.

The necessity for defining the condition of zero phase difference occurs only in the case of orthogonal vibrations because one cannot be "resolved" into the other. When P_1 and P_2 are not orthogonal, then the resolved component of one along the other can be compared for specifying their phase difference. The resultant intensity is then a maximum, when the phase difference is zero as seen from Eq. (4.4), and the resultant state of polarisation lies on the arc $P_1 P_2$ directly joining P_1 and P_2. When the two beams are opposite in phase, the intensity is a minimum and the resultant state lies on the greater segment $(P_1 P_a P_2)$ of the great circle through P_1 and P_2.

It follows from Eqs. (4.5) and (4.6) that, when the phase difference between the two beams is altered without altering the ratio of their intensities, then $\sin^2 a/\sin^2 b$ is a constant. The locus of P is then a small circle, with its centre on the great circle through P_1 and P_2 (Fig. 6b). On the other hand, if the ratio of the intensities is altered, keeping the phase difference constant, then ε' is a constant, and the locus of P is again a small circle, but passing through P_1 and P_2, with its centre of the great circle which is the perpendicular bisector of the arc $P_1 P_2$ (Fig. 6b). When P_1 and P_2 are orthogonal, the former family of small circles are all perpendicular to the diameter $P_1 P_2$ and the latter all become great circles passing through P_1 and P_2 (Fig. 6a).

β) Interference of two beams after resolution by an analyser. Given a vibration in state P_1 (Fig. 5) its resolved components in the orthogonal states P and P_a

can be said to be in phase by choosing the arc PP_2P_a as the standard arc defining the zero of phase difference for two orthogonal states. Considering a second vibration in state P_2, let us also resolve it into its components in the states P and P_a. Let δ' be the phase advance of the P-component of the vibration in state P_2 over the P-component of the vibration in state P_1; and similarly let δ'' be the difference in the phases of the P_a-components of the vibrations in states P_2 and P_1 respectively. Then from a consideration of the results of the preceding sub-section,

$$\delta' - \delta'' = \hat{P} \tag{4.15}$$

where $\hat{P}$ is the angle $P_1\hat{P}P_2$, counted positive if (on looking from P to P_a) an anticlockwise rotation brings arc PP_1 to arc PP_2.

The result of the last paragraph may be used to discuss a problem of common occurrence in crystal optics (see e.g. Chap. C). Two beams 1 and 2 initially of intensities I_1 and I_2 and in states of polarisation P_1 and P_2—the first having a phase advance δ over the second—are made to interfere after transmission through an analyser which resolves them to the same state of polarisation P. (Note that in the present context P does *not* represent the resultant state obtained by directly compounding the beams 1 and 2.) The P-components of the beams of polarisation P_1 and P_2 will have intensities $I_1 \cos^2 b$ and $I_2 \cos^2 a$ respectively and our main problem in this section is to determine their phase difference δ'. The intensity transmitted by an analyser P is then given by

$$I_P = I_1 \cos^2 b + I_2 \cos^2 a + 2\sqrt{I_1 I_2} \cos a \cos b \cos \delta'. \tag{4.16}$$

Similarly the P_a-component of the resultant beam will have an intensity

$$I_{P_a} = I_1 \sin^2 b + I_2 \sin^2 a + 2\sqrt{I_1 I_2} \sin a \sin b \cos \delta''. \tag{4.17}$$

The intensity I of the resultant beam, obtained by directly compounding 1 and 2, is obtained by adding (4.16) and (4.17) using (4.15):

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\{\cos a \cos b \cos \delta' + \sin a \sin b \cos(\delta' - \hat{P})\}.$$

By applying the standard expressions for the spherical excess of a triangle this reduces to

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos c \cos(\delta' + \tfrac{1}{2}\varepsilon) \tag{4.18}$$

where ε represents the area or spherical excess of the triangle PP_1P_2 itself (counted positive if the sequence of points P, P_1, P_2 describe the periphery of the triangle in a counter-clockwise sense).

Comparing (4.18) with (4.4) we obtain the interesting result that if two beams initially have a phase difference δ then after passage through an analyser their phase difference becomes

$$\delta' = \delta - \tfrac{1}{2}\varepsilon, \tag{4.19}$$

i.e., an additional phase difference $-\tfrac{1}{2}\varepsilon$ is introduced in the process of analysation. The intensity transmitted by the analyser (i.e., the intensity obtained by the interference of the resolved components) is obtained by substituting (4.19) in (4.16):

$$I_P = I_1 \cos^2 b + I_2 \cos^2 a + 2\sqrt{I_1 I_2} \cos a \cos b \cos(\delta - \tfrac{1}{2}\varepsilon). \tag{4.20}$$

The limiting case when the states of polarisation P_1 and P_2 become oppositely polarised is of particular importance (Fig. 6a). In this case, if the beams have been originally derived by the decomposition of a beam in state P', we must take the great circle $P_1 P' P_2$ as defining the condition of zero phase difference. It follows from (4.19) (since ε becomes now the area of a lune) that on passing through an analyser P the resolved component of the first beam lags behind that of the second by an angle Δ which denotes the angle $P P_1 P'$ (measured positive in a counter-clockwise sense). Thus, for example, when two circularly polarised beams in opposite states are incident on a linear (or elliptic) analyser, the phase difference between the transmitted beams is altered by 2ϑ when the azimuth of the analyser is rotated (as a whole) through an angle ϑ—a result which finds application in certain types of phase-contrast microscopes which use crystal-optic elements[1].

5. Propagation of light through an optical system (no absorption). *α) Non-absorbing optical elements of infinitesimal thickness.* We wish to investigate the change in the state of polarisation of a beam of light of polarisation state P as a result of its passing through a number of optical elements. Each element is considered to be either (a) a parallel plate of birefringent material, with principal planes oriented at an arbitrary azimuth, or (b) an optically active material, which only rotates the azimuth of the elliptically polarised beam. Systems of this type were considered by JONES[2] making use of a matrix calculus and his papers may be referred to for examples and for further details. The matrix method of JONES is also discussed in Sect. 12. The overall effect can however be readily worked out by the use of the Poincaré sphere.

Before proceeding to the general case we shall first consider a special case of such combination, which is of particular interest, viz., when the effect of each optical element is infinitesimal in magnitude. An example is that of a birefringent optically active crystal. Although strictly the medium must be considered to have the properties of both birefringence and optical activity and should be treated as such in a rigorous theory (see Chap. B), one may also picture the crystal to be made up of alternate infinitesimal layers of equal thickness exhibiting alternately, only linear birefringence and only optical activity. A thickness dz of the optically active birefringent medium can on the above picture be regarded as a linearly birefringent element producing a retardation $d\delta = \delta' \, dz$, and an optically active element producing a rotation $d\varrho = \varrho' \, dz$ where δ' and ϱ' define respectively the retardation per unit thickness in the absence of optical activity and the optical rotatory power in the absence of linear birefringence. Suppose the principal axes of the birefringent element are at azimuth α and $\alpha + \pi/2$, represented by M and N (Fig. 7) of which M is the faster axis. Then the effect of passage through these two optical elements is to rotate the Poincaré sphere through angles $d\delta$ and $2d\varrho$ in an anti-clockwise direction about MN and LR respectively (Fig. 7). The addition of two infinitesimal rotations follow the law of vectorial addition and the resultant is independent of the sequence and is a rotation through an angle $d\Delta = \sqrt{(d\delta)^2 + (2d\varrho)^2}$ about the axis EF which is in the plane of MN and LR and makes an angle 2χ with NM where

$$2\chi = \operatorname{arc\,tan} \frac{2d\varrho}{d\delta} = \operatorname{arc\,tan} \frac{2\varrho'}{\delta'} . \tag{5.1}$$

For unit thickness of a birefringent, optically active crystal, the resultant effect is an anti-clockwise rotation of the Poincaré sphere through an angle

$$\Delta' = \sqrt{\delta'^2 + (2\varrho')^2} \tag{5.2}$$

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[1] See e.g. BENNETT, OSTERBERG, JUPNIK and RICHARDS: Phase Microscopy, Chap. 3. New York 1951.

[2] R.C. JONES: J. Opt. Soc. Amer. **31**, 488, 493, 500 (1941).

about the axis EF, where the elliptic state E is propagated with the faster velocity.

Thus, the most general type of non-absorbing crystal (or optical element) is one which leads to a rotation of the Poincaré sphere about an axis EF, which is neither the polar axis LR nor does it lie in the equatorial plane. Analogous to the purely birefringent crystal, in which linear vibrations parallel to its principal directions are propagated unchanged, and the purely optically active crystal without birefringence, in which L and R are propagated unchanged, light of polarisation states E and F will be propagated unchanged in this crystal. This is so because a rotation of the sphere about EF leaves E and F unchanged. These states are two crossed ellipses which are orthogonal to each other.

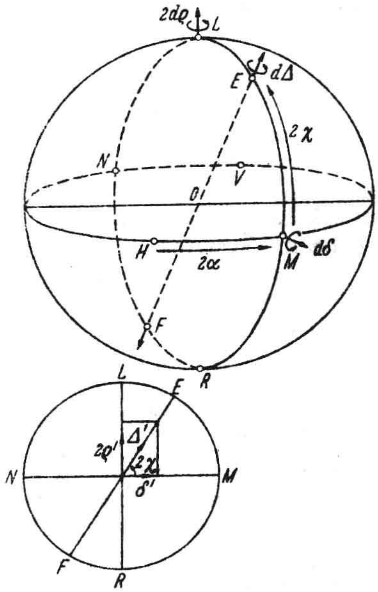

Fig. 7. Effect of a non-absorbing crystal exhibiting birefringence and optical activity. If δ' is the phase difference due to birefringence alone and ϱ' the optical rotation in the absence of birefringence, the resultant effect is a rotation of the Poincaré sphere through an angle Δ' about the axis EF.

In such a crystal, incident light of arbitrary state of polarisation P_0 is split up into the two orthogonal elliptical states E and F, which are propagated unchanged in state, but with a relative phase retardation Δ' per unit thickness. On emergence, they recombine, and the resultant state P is obtained from P_0 by a rotation of the Poincaré sphere about the axis EF, as shown in Sect. 4. The optical phenomena in such crystals are treated in Chap. B.

Since the emerging waves are orthogonally polarised they do not interfere and the emergent intensity will be the same as the incident intensity. The crystal will therefore be transparent as is to be expected. Vice versa, the operation for a thin layer of any non-absorbing optical element must necessarily be a rotation through an infinitesimal angle $d\Delta = \Delta' dz$ about some axis EF. This can be resolved into three infinitesimal rotations $d\Delta_1$, $d\Delta_2$, $d\Delta_3$ about the axes HV, CD, and LR respectively. These axes correspond to the co-ordinate axes OU, OV, OW in Poincaré space (Fig. 2). Thus, the effect of a general infinitesimal (non-absorbing) optical element on the state of polarisation of light passing through it is describable by means of three infinitesimal rotations about OU, OV and OW.

β) Combined effect of a series of transparent plates. We now return to the problem stated at the beginning of the section, viz., the passage of polarised light through a series of transparent parallel plates of finite thickness. For a linearly birefringent plate producing a relative phase retardation δ, the effect is to rotate the Poincaré sphere about an axis in the equatorial plane through the angle δ. The orientation of the axis is known from the orientation of the principal plane. So also, if ϱ is the rotation produced by the optically active plate (ϱ is positive for left-rotation), then the effect is to rotate the sphere through an angle 2ϱ about LR. (If the system also contains plates possessing both linear birefringence and optical activity, the effect of any such plate is to rotate the sphere about a given axis EF through a given angle Δ.)

The resultant of two successive rotations about two axes is again a rotation about some other axis of the sphere which may be determined either analytically or graphically by the construction illustrated in Fig. 8. The combined effects