国外数学名著系列

（影印版）30

Peter Petersen

Riemannian Geometry

Second Edition

黎曼几何

（第二版）

科学出版社
www.sciencep.com

国外数学名著系列(影印版) 30

Riemannian Geometry

Second Edition

黎曼几何

(第二版)

Peter Petersen

科学出版社
北京

图字:01-2006-7388

Peter Petersen: Riemannian Geometry (Second Edition)

图书在版编目(CIP)数据

黎曼几何:第2版=Riemannian Geometry (Second Edition)/(美)彼得森(Petersen, P.)著.—影印版.—北京:科学出版社,2007
(国外数学名著系列)

ISBN 978-7-03-018294-4

Ⅰ.黎… Ⅱ.彼… Ⅲ.黎曼几何-英文 Ⅳ.O186.12

中国版本图书馆 CIP 数据核字(2006)第 153196 号

责任编辑:范庆奎/责任印刷:安春生/封面设计:黄华斌

科学出版社 出版
北京东黄城根北街16号
邮政编码:100717
http://www.sciencep.com

中国科学院印刷厂 印刷

科学出版社发行 各地新华书店经销

*

2007年1月第 一 版 开本:B5(720×1000)
2007年1月第一次印刷 印张:26 1/2
印数:1—3 500 字数:494 000

定价:76.00元

(如有印装质量问题,我社负责调换(科印))

中国科学院科学出版基金资助出版

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

To my wife, Laura

Preface

This book is meant to be an introduction to Riemannian geometry. The reader is assumed to have some knowledge of standard manifold theory, including basic theory of tensors, forms, and Lie groups. At times we shall also assume familiarity with algebraic topology and de Rham cohomology. Specifically, we recommend that the reader is familiar with texts like [**14**], [**63**], or [**87**, vol. 1]. For the readers who have only learned a minimum of tensor analysis we have an appendix which covers Lie derivatives, forms, Stokes' theorem, Čech cohomology, and de Rham cohomology. The reader should also have a nodding acquaintance with ordinary differential equations. For this, a text like [**67**]is more than sufficient.

Most of the material usually taught in basic Riemannian geometry, as well as several more advanced topics, is presented in this text. Several theorems from chapters 7 to 11 appear for the first time in textbook form. This is particularly surprising as we have included essentially only the material students of Riemannian geometry must know.

The approach we have taken sometimes deviates from the standard path. Aside from the usual variational approach (added in the second edition) we have also developed a more elementary approach that simply uses standard calculus together with some techniques from differential equations. Our motivation for this treatment has been that examples become a natural and integral part of the text rather than a separate item that is sometimes minimized. Another desirable by-product has been that one actually gets the feeling that gradients, Hessians, Laplacians, curvatures, and many other things are actually computable.

We emphasize throughout the text the importance of using the correct type of coordinates depending on the theoretical situation at hand. First, we develop a substitute for the second variation formula by using adapted frames or coordinates. This is the approach mentioned above that can be used as an alternative to variational calculus. These are coordinates naturally associated to a distance function. If, for example we use the function that measures the distance to a point, then the adapted coordinates are nothing but polar coordinates. Next, we have exponential coordinates, which are of fundamental importance in showing that distance functions are smooth. Then distance coordinates are used first to show that distance-preserving maps are smooth, and then later to give good coordinate systems in which the metric is sufficiently controlled so that one can prove, say, Cheeger's finiteness theorem. Finally, we have harmonic coordinates. These coordinates have some magical properties. One, in particular, is that in such coordinates the Ricci curvature is essentially the Laplacian of the metric.

From a more physical viewpoint, the reader will get the idea that we are also using the Hamilton-Jacobi equations instead of only relying on the Euler-Lagrange equations to develop Riemannian geometry (see [5]for an explanation of these matters). It is simply a matter of taste which path one wishes to follow, but surprisingly, the Hamilton-Jacobi approach has never been tried systematically in Riemannian geometry.

The book can be divided into five imaginary parts

Part I: Tensor geometry, consisting of chapters 1-4.

Part II: Classical geodesic geometry, consisting of chapters 5 and 6.

Part III: Geometry à la Bochner and Cartan, consisting of chapters 7 and 8.

Part IV: Comparison geometry, consisting of chapters 9-11.

Appendix: De Rham cohomology.

Chapters 1-8 give a pretty complete picture of some of the most classical results in Riemannian geometry, while chapters 9-11 explain some of the more recent developments in Riemannian geometry. The individual chapters contain the following material:

Chapter 1: Riemannian manifolds, isometries, immersions, and submersions are defined. Homogeneous spaces and covering maps are also briefly mentioned. We have a discussion on various types of warped products, leading to an elementary account of why the Hopf fibration is also a Riemannian submersion.

Chapter 2: Many of the tensor constructions one needs on Riemannian manifolds are developed. First the Riemannian connection is defined, and it is shown how one can use the connection to define the classical notions of Hessian, Laplacian, and divergence on Riemannian manifolds. We proceed to define all of the important curvature concepts and discuss a few simple properties. Aside from these important tensor concepts, we also develop several important formulas that relate curvature and the underlying metric. These formulas are to some extent our replacement for the second variation formula. The chapter ends with a short section where such tensor operations as contractions, type changes, and inner products are briefly discussed.

Chapter 3: First, we indicate some general situations where it is possible to diagonalize the curvature operator and Ricci tensor. The rest of the chapter is devoted to calculating curvatures in several concrete situations such as: spheres, product spheres, warped products, and doubly warped products. This is used to exhibit some interesting examples that are Ricci flat and scalar flat. In particular, we explain how the Riemannian analogue of the Schwarzschild metric can be constructed. Several different models of hyperbolic spaces are mentioned. We have a section on Lie groups. Here two important examples of left-invariant metrics are discussed as well the general formulas for the curvatures of bi-invariant metrics. Finally, we explain how submersions can be used to create new examples. We have paid detailed attention to the complex projective space. There are also some general comments on how submersions can be constructed using isometric group actions.

Chapter 4: Here we concentrate on the special case where the Riemannian manifold is a hypersurface in Euclidean space. In this situation, one gets some special relations between curvatures. We give examples of simple Riemannian manifolds that cannot be represented as hypersurface metrics. Finally we give a brief introduction to the global Gauss-Bonnet theorem and its generalization to higher dimensions.

Chapter 5: This chapter further develops the foundational topics for Riemannian manifolds. These include, the first variation formula, geodesics, Riemannian manifolds as metric spaces, exponential maps, geodesic completeness versus metric completeness, and maximal domains on which the exponential map is an embedding. The chapter ends with the classification of simply connected space forms and metric characterizations of Riemannian isometries and submersions.

Chapter 6: We cover two more foundational techniques: parallel translation and the second variation formula. Some of the classical results we prove here are: The Hadamard-Cartan theorem, Cartan's center of mass construction in nonpositive curvature and why it shows that the fundamental group of such spaces are torsion free, Preissmann's theorem, Bonnet's diameter estimate, and Synge's lemma. We have supplied two proofs for some of the results dealing with non-positive curvature in order that people can see the difference between using the variational (or Euler-Lagrange) method and the Hamilton-Jacobi method. At the end of the chapter we explain some of the ingredients needed for the classical quarter pinched sphere theorem as well as Berger's proof of this theorem. Sphere theorems will also be revisited in chapter 11.

Chapter 7: Many of the classical and more recent results that arise from the Bochner technique are explained. We start with Killing fields and harmonic 1-forms as Bochner did, and finally, discuss some generalizations to harmonic p-forms. For the more advanced audience we have developed the language of Clifford multiplication for the study p-forms, as we feel that it is an important way of treating this material. The last section contains some more exotic, but important, situations where the Bochner technique is applied to the curvature tensor. These last two sections can easily be skipped in a more elementary course. The Bochner technique gives many nice bounds on the topology of closed manifolds with nonnegative curvature. In the spirit of comparison geometry, we show how Betti numbers of nonnegatively curved spaces are bounded by the prototypical compact flat manifold: the torus.

The importance of the Bochner technique in Riemannian geometry cannot be sufficiently emphasized. It seems that time and again, when people least expect it, new important developments come out of this simple philosophy.

While perhaps only marginally related to the Bochner technique we have also added a discussion on how the presence of Killing fields in positive sectional curvature can lead to topological restrictions. This is a rather new area in Riemannian geometry that has only been developed in the last 15 years.

Chapter 8: Part of the theory of symmetric spaces and holonomy is developed. The standard representations of symmetric spaces as homogeneous spaces and via Lie algebras are explained. We prove Cartan's existence theorem for isometries. We explain how one can compute curvatures in general and make some concrete calculations on several of the Grassmann manifolds including complex projective space. Having done this, we define holonomy for general manifolds, and discuss the de Rham decomposition theorem and several corollaries of it. The above examples are used to give an idea of how one can classify symmetric spaces. Also, we show in the same spirit why symmetric spaces of (non)compact type have (nonpositive) nonnegative curvature operator. Finally, we present a brief overview of how holonomy and symmetric spaces are related with the classification of holonomy groups. This is used in a grand synthesis, with all that has been learned up to this point, to give Gallot and Meyer's classification of compact manifolds with nonnegative curvature operator.

Chapter 9: Manifolds with lower Ricci curvature bounds are investigated in further detail. First, we discuss volume comparison and its uses for Cheng's maximal diameter theorem. Then we investigate some interesting relationships between Ricci curvature and fundamental groups. The strong maximum principle for continuous functions is developed. This result is first used in a warm-up exercise to give a simple proof of Cheng's maximal diameter theorem. We then proceed to prove the Cheeger-Gromoll splitting theorem and discuss its consequences for manifolds with nonnegative Ricci curvature.

Chapter 10: Convergence theory is the main focus of this chapter. First, we introduce the weakest form of convergence: Gromov-Hausdorff convergence. This concept is often useful in many contexts as a way of getting a weak form of convergence. The real object is then to figure out what weak convergence implies, given some stronger side conditions. There is a section which breezes through Hölder spaces, Schauder's elliptic estimates and harmonic coordinates. To facilitate the treatment of the stronger convergence ideas, we have introduced a norm concept for Riemannian manifolds. We hope that these norms will make the subject a little more digestible. The main idea of this chapter is to prove the Cheeger-Gromov convergence theorem, which is called the Convergence Theorem of Riemannian Geometry, and Anderson's generalizations of this theorem to manifolds with bounded Ricci curvature.

Chapter 11: In this chapter we prove some of the more general finiteness theorems that do not fall into the philosophy developed in chapter 10. To begin, we discuss generalized critical point theory and Toponogov's theorem. These two techniques are used throughout the chapter to prove all of the important theorems. First, we probe the mysteries of sphere theorems. These results, while often unappreciated by a larger audience, have been instrumental in developing most of the new ideas in the subject. Comparison theory, injectivity radius estimates, and Toponogov's theorem were first used in a highly nontrivial way to prove the classical quarter pinched sphere theorem of Rauch, Berger, and Klingenberg. Critical point theory was invented by Grove and Shiohama to prove the diameter sphere theorem. After the sphere theorems, we go through some of the major results of comparison geometry: Gromov's Betti number estimate, The Soul theorem of Cheeger and Gromoll, and The Grove-Petersen homotopy finiteness theorem.

Appendix A: Here, some of the important facts about forms and tensors are collected. Since Lie derivatives are used rather heavily at times we have included an initial section on this. Stokes' theorem is proved, and we give a very short and streamlined introduction to Čech and de Rham cohomology. The exposition starts with the assumption that we only work with manifolds that can be covered by finitely many charts where all possible intersections are contractible. This makes it very easy to prove all of the major results, as one can simply use the Poincaré and Meyer-Vietoris lemmas together with induction on the number of charts in the covering.

At the end of each chapter, we give a list of books and papers that cover and often expand on the material in the chapter. We have whenever possible attempted to refer just to books and survey articles. The reader is then invited to go from those sources back to the original papers. For more recent works, we also give journal references if the corresponding books or surveys do not cover all aspects of the original paper. One particularly exhaustive treatment of Riemannian Geometry for the reader who is interested in learning more is [**11**]. Other valuable texts that expand or complement much of the material covered here are [**70**], [**87**]and [**90**]. There is also a historical survey by Berger (see [**10**]) that complements this text very well.

A first course should definitely cover chapters 2, 5, and 6 together with whatever one feels is necessary from chapters 1, 3, and 4. Note that chapter 4 is really a world unto itself and is not used in a serious way later in the text. A more advanced course could consist of going through either part III or IV as defined earlier. These parts do not depend in a serious way on each other. One can probably not cover the entire book in two semesters, but one can cover parts I, II, and III or alternatively I, II, and IV depending on one's inclination. It should also be noted that, if one ignores the section on Killing fields in chapter 7, then this material can actually be covered without having been through chapters 5 and 6. Each of the chapters ends with a collection of exercises. These exercises are designed both to reinforce the material covered and to establish some simple results that will be needed later. The reader should at least read and think about all of the exercises, if not actually solve all of them.

There are several people I would like to thank. First and foremost are those students who suffered through my various pedagogical experiments with the teaching of Riemannian geometry. Special thanks go to Marcel Berger, Hao Fang, Semion Shteingold, Chad Sprouse, Marc Troyanov, Gerard Walschap, Nik Weaver, Fred Wilhelm and Hung-Hsi Wu for their constructive criticism of parts of the book. For the second edition I'd also like to thank Edward Fan, Ilkka Holopainen, Geoffrey Mess, Yanir Rubinstein, and Burkhard Wilking for making me aware of typos and other deficiencies in the first edition. I would especially like to thank Joseph Borzellino for his very careful reading of this text, and Peter Blomgren for writing the programs that generated Figures 2.1 and 2.2. Finally I would like to thank Robert Greene, Karsten Grove, and Gregory Kallo for all the discussions on geometry we have had over the years.

The author was supported in part by NSF grants DMS 0204177 and DMS 9971045.

《国外数学名著系列》(影印版)

（按初版出版时间排序）

1. 拓扑学Ⅰ：总论　S. P. Novikov(Ed.)　2006.1
2. 代数学基础　Igor R. Shafarevich　2006.1
3. 现代数论导引(第二版)　Yu. I. Manin　A. A. Panchishkin　2006.1
4. 现代概率论基础(第二版)　Olav Kallenberg　2006.1
5. 数值数学　Alfio Quarteroni　Riccardo Sacco　Fausto Saleri　2006.1
6. 数值最优化　Jorge Nocedal　Stephen J. Wright　2006.1
7. 动力系统　Jürgen Jost　2006.1
8. 复杂性理论　Ingo Wegener　2006.1
9. 计算流体力学原理　Pieter Wesseling　2006.1
10. 计算统计学基础　James E. Gentle　2006.1
11. 非线性时间序列　Jianqing Fan　Qiwei Yao　2006.1
12. 函数型数据分析(第二版)　J. O. Ramsay　B. W. Silverman　2006.1
13. 矩阵迭代分析(第二版)　Richard S. Varga　2006.1
14. 偏微分方程的并行算法　Petter Bjørstad　Mitchell Luskin (Eds.)　2006.1
15. 非线性问题的牛顿法　Peter Deuflhard　2006.1
16. 区域分解算法：算法与理论　A. Toselli　O. Widlund　2006.1
17. 常微分方程的解法Ⅰ：非刚性问题(第二版)　E. Hairer　S. P. Nørsett　G. Wanner　2006.1
18. 常微分方程的解法Ⅱ：刚性与微分代数问题(第二版)　E. Hairer　G. Wanner　2006.1
19. 偏微分方程与数值方法　Stig Larsson　Vidar Thomée　2006.1
20. 椭圆型微分方程的理论与数值处理　W. Hackbusch　2006.1
21. 几何拓扑：局部性、周期性和伽罗瓦对称性　Dennis P. Sullivan　2006.1
22. 图论编程：分类树算法　Victor N. Kasyanov　Vladimir A. Evstigneev　2006.1
23. 经济、生态与环境科学中的数学模型　Natali Hritonenko　Yuri Yatsenko　2006.1

24. 代数数论　Jürgen Neukirch　2007.1
25. 代数复杂性理论　Peter Bürgisser　Michael Clausen　M. Amin Shokrollahi　2007.1
26. 一致双曲性之外的动力学：一种整体的几何学的与概率论的观点　Christian Bonatti　Lorenzo J. Díaz　Marcelo Viana　2007.1
27. 算子代数理论Ⅰ　Masamichi Takesaki　2007.1
28. 离散几何中的研究问题　Peter Brass　William Moser　János Pach　2007.1
29. 数论中未解决的问题(第三版)　Richard K. Guy　2007.1

30. 黎曼几何(第二版)　Peter Petersen　2007. 1
31. 递归可枚举集和图灵度:可计算函数与可计算生成集研究　Robert I. Soare　2007. 1
32. 模型论引论　David Marker　2007. 1
33. 线性微分方程的伽罗瓦理论　Marius van der Put　Michael F. Singer　2007. 1
34. 代数几何 II:代数簇的上同调,代数曲面　I. R. Shafarevich(Ed.)　2007. 1
35. 伯克利数学问题集(第三版)　Paulo Ney de Souza　Jorge-Nuno Silva　2007. 1
36. 陶伯理论:百年进展　Jacob Korevaar　2007. 1

Contents

Preface

Chapter 1. Riemannian Metrics 1
1. Riemannian Manifolds and Maps 2
2. Groups and Riemannian Manifolds 5
3. Local Representations of Metrics 8
4. Doubly Warped Products 13
5. Exercises 17

Chapter 2. Curvature 21
1. Connections 22
2. The Connection in Local Coordinates 29
3. Curvature 32
4. The Fundamental Curvature Equations 41
5. The Equations of Riemannian Geometry 47
6. Some Tensor Concepts 51
7. Further Study 56
8. Exercises 56

Chapter 3. Examples 63
1. Computational Simplifications 63
2. Warped Products 64
3. Hyperbolic Space 74
4. Metrics on Lie Groups 77
5. Riemannian Submersions 82
6. Further Study 90
7. Exercises 90

Chapter 4. Hypersurfaces 95
1. The Gauss Map 95
2. Existence of Hypersurfaces 97
3. The Gauss-Bonnet Theorem 101
4. Further Study 107
5. Exercises 108

Chapter 5. Geodesics and Distance 111
1. Mixed Partials 112
2. Geodesics 116
3. The Metric Structure of a Riemannian Manifold 121
4. First Variation of Energy 126
5. The Exponential Map 130

6. Why Short Geodesics Are Segments 132
7. Local Geometry in Constant Curvature 134
8. Completeness 137
9. Characterization of Segments 139
10. Riemannian Isometries 143
11. Further Study 149
12. Exercises 149

Chapter 6. Sectional Curvature Comparison I 153
1. The Connection Along Curves 153
2. Second Variation of Energy 158
3. Nonpositive Sectional Curvature 162
4. Positive Curvature 169
5. Basic Comparison Estimates 173
6. More on Positive Curvature 176
7. Further Study 182
8. Exercises 183

Chapter 7. The Bochner Technique 187
1. Killing Fields 188
2. Hodge Theory 202
3. Harmonic Forms 205
4. Clifford Multiplication on Forms 213
5. The Curvature Tensor 221
6. Further Study 229
7. Exercises 229

Chapter 8. Symmetric Spaces and Holonomy 235
1. Symmetric Spaces 236
2. Examples of Symmetric Spaces 244
3. Holonomy 252
4. Curvature and Holonomy 256
5. Further Study 262
6. Exercises 263

Chapter 9. Ricci Curvature Comparison 265
1. Volume Comparison 265
2. Fundamental Groups and Ricci Curvature 273
3. Manifolds of Nonnegative Ricci Curvature 279
4. Further Study 290
5. Exercises 290

Chapter 10. Convergence 293
1. Gromov-Hausdorff Convergence 294
2. Hölder Spaces and Schauder Estimates 301
3. Norms and Convergence of Manifolds 307
4. Geometric Applications 318
5. Harmonic Norms and Ricci curvature 321
6. Further Study 330
7. Exercises 331

Chapter 11. Sectional Curvature Comparison II 333
1. Critical Point Theory 333
2. Distance Comparison 338
3. Sphere Theorems 346
4. The Soul Theorem 349
5. Finiteness of Betti Numbers 357
6. Homotopy Finiteness 365
7. Further Study 372
8. Exercises 372

Appendix. De Rham Cohomology 375
1. Lie Derivatives 375
2. Elementary Properties 379
3. Integration of Forms 380
4. Čech Cohomology 383
5. De Rham Cohomology 384
6. Poincaré Duality 387
7. Degree Theory 389
8. Further Study 391

Bibliography 393

Index 397