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Constantine M. Dafermos 著

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# Preface

The seeds of Continuum Physics were planted with the works of the natural philosophers of the eighteenth century, most notably Euler; by the mid-nineteenth century, the trees were fully grown and ready to yield fruit. It was in this environment that the study of gas dynamics gave birth to the theory of quasilinear hyperbolic systems in divergence form, commonly called "hyperbolic conservation laws"; and these two subjects have been traveling hand-in-hand over the past one hundred and fifty years. This book aims at presenting the theory of hyperbolic conservation laws from the standpoint of its genetic relation to Continuum Physics. Even though research is still marching at a brisk pace, both fields have attained by now the degree of maturity that would warrant the writing of such an exposition.

In the realm of Continuum Physics, material bodies are realized as continuous media, and so-called "extensive quantities", such as mass, momentum and energy, are monitored through the fields of their densities, which are related by balance laws and constitutive equations. A self-contained, though skeletal, introduction to this branch of classical physics is presented in Chapter II. The reader may flesh it out with the help of a specialized text on the subject.

In its primal formulation, the typical balance law stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of the body is balanced by the rate of flux of this quantity through the boundary of the subdomain together with the rate of its production inside the subdomain. In the absence of production, a balanced extensive quantity is conserved. The special feature that renders Continuum Physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, i.e., partial differential equations in divergence form.

The collection of balance laws in force demarcates and identifies particular continuum theories, such as Mechanics, Thermomechanics, Electrodynamics and so on. In the context of a continuum theory, constitutive equations specify the nature of the medium, for example viscous fluid, elastic solid, elastic dielectric, etc. In conjunction with these constitutive relations, the field equations yield closed systems of partial differential equations, dubbed "balance laws" or "conservation laws", from which the equilibrium state or motion of the continuous medium is to be determined. Historically, the vast majority of noteworthy partial differential equations were generated through that process. The central thesis of this book is that the umbilical cord joining Continuum Physics with the theory of partial differential equations should not be severed, as it is still carrying nourishment in both directions.

Systems of balance laws may be elliptic, typically in statics; hyperbolic, in dynamics, for media with "elastic" response; mixed elliptic-hyperbolic, in statics or dynamics, when the medium undergoes phase transitions; parabolic or mixed parabolic-hyperbolic, in the presence of viscosity, heat conductivity or other diffusive mechanisms. Accordingly, the basic notions shall be introduced, in Chapter I, at a level of generality that would encompass all of the above possibilities. Nevertheless, since the subject of this work is hyperbolic conservation laws, the discussion will eventually focus on such systems, beginning with Chapter III.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profiles of compression waves get progressively steeper and eventually break, generating jump discontinuities which propagate on as shocks. Hence, inevitably, the theory must deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. It thus becomes necessary to devise proper criteria for singling out admissible weak solutions. Continuum Physics naturally induces such admissibility criteria through the Second Law of thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. The notions of "entropy" and "vanishing diffusion", which will play a central rôle throughout the book, are first introduced in Chapters III and IV.

From the standpoint of analysis, a very elegant, definitive theory is available for the case of scalar conservation laws, in one or several space dimensions, which is presented in detail in Chapter VI. By contrast, systems of conservation laws in several space dimensions are still terra incognita, as the analysis is currently facing insurmountable obstacles. The relatively modest results derived thus far, pertaining to local existence and stability of smooth or piecewise smooth solutions, underscore the importance of the special structure of the field equations of Continuum Physics and the stabilizing role of the Second Law of thermodynamics. These issues are discussed in Chapter V.

Beginning with Chapter VII, the focus of the investigation is fixed on systems of conservation laws in one-space dimension. In that setting, the theory has a number of special features, which are of great help to the analyst, so major progress has been achieved.

Chapter VIII provides a systematic exposition of the properties of shocks. In particular, various shock admissibility criteria are introduced, compared and contrasted. Admissible shocks are then combined, in Chapter IX, with another class of particular solutions, called centered rarefaction waves, to synthesize wave fans that solve the classical Riemann problem. Solutions of the Riemann problem may in turn be employed as building blocks for constructing solutions to the Cauchy problem, in the class $BV$ of functions of bounded variation. For that purpose, two construction methods will be presented here: The random choice scheme, in Chapter XIII, and a front tracking algorithm, in Chapter XIV. Uniqueness and stability of these solutions will also be established. The main limitation of this approach is that it generally applies only when the initial data have sufficiently small total variation. This restriction seems to be generally necessary, as it turns out that, in certain systems, when the initial data are "large" even weak solutions to the Cauchy problem may blow up in finite time. However, whether such catastrophes may occur to solutions of the field equations of Continuum Physics is at present a major open problem.

There are other interesting properties of weak solutions, beyond existence and uniqueness. In Chapter X, the notion of characteristic is extended from classical to weak solutions and is employed for obtaining a very precise description of regularity and long time behavior of solutions to scalar conservation laws, in Chapter XI, as well as to systems of two conservation laws, in Chapter XII.

Finally, Chapter XV introduces the concept of measure-valued solution and outlines the functional analytic method of compensated compactness, which determines solutions to hyperbolic systems of conservation laws as weak limits of sequences of approximate solutions, constructed via a variety of approximating schemes.

In order to highlight the fundamental ideas, the discussion proceeds from the general to the particular, notwithstanding the clear pedagogical advantage of the reverse course. Moreover, the pace of the proofs is purposely uneven: slow for the basic, elementary propositions that may provide material for an introductory course; faster for the more advanced technical results that are addressed to the experienced analyst. Even though the various parts of this work fit together to form an integral entity, readers may select a number of independent itineraries through the book. Thus, those principally interested in the conceptual foundations of the theory of hyperbolic conservation laws, in connection to Continuum Physics, need only go through Chapters I–V. Chapter VI, on the scalar conservation law, may be read virtually independently of the rest. Students intending to study solutions as compositions of interacting elementary waves may begin with Chapters VII–IX and then either continue on to Chapters X–XII or else pass directly to Chapter XIII and/or Chapter XIV. Finally, only Chapter VII is needed as a prerequisite for the functional analytic approach expounded in Chapter XV.

Twenty-five years ago, it might have been feasible to write a treatise surveying the entire area; however, the explosive development of the subject over the past several years has rendered such a goal unattainable. Thus, even though this work strives to present a panoramic view of the terrain, certain noteworthy features had to be left out. The most conspicuous absence is a discussion of numerics. This is regrettable, because, beyond its potential practical applications, the numerical analysis of hyperbolic conservation laws provides valuable insight to the theory. Fortunately, a number of specialized texts on that subject are currently available. Several other important topics receive only superficial treatment here, so the reader may have to resort to the cited references for a more thorough investigation. On the other hand, certain topics are perhaps discussed in excessive detail, as they are of special interest to the author. A number of results are published here for the first time. Though extensive, the bibliography is far from exhaustive. In any case, the whole subject is in a state of active development, and significant new publications appear with considerable frequency.

My teachers, Jerry Ericksen and Clifford Truesdell, initiated me to Continuum Physics, as living scientific subject and as formal mathematical structure with fascinating history. I trust that both views are somehow reflected in this work.

I am grateful to many scientists – teachers, colleagues and students alike – who have helped me, over the past thirty years, to learn Continuum Physics and the theory of hyperbolic conservation laws. Since it would be impossible to list them all here by name, let me single out Stu Antman, John Ball, Alberto Bressan, Gui-Qiang Chen, Bernie Coleman, Ron DiPerna, Jim Glimm, Jim Greenberg, Mort Gurtin, Ling Hsiao, Barbara Keyfitz, Peter Lax, Philippe LeFloch, Tai-Ping Liu, Andy Majda, Piero Marcati, Walter Noll, Denis Serre, Marshal Slemrod, Luc Tartar, Konstantina Trivisa, Thanos Tzavaras and Zhouping Xin, who have also honored me with their friendship. In particular, Denis Serre's persistent encouragement helped me to carry this arduous project to completion.

The frontispiece figure depicts the intricate wave pattern generated by shock reflections in the supersonic gas flow through a Laval nozzle with wall disturbances. This beautiful interferogram, brought to my attention by John Ockendon, was produced by W.J. Hiller and G.E.A. Meier at the Max-Planck-Institut für Strömungsforschung, in Göttingen. It is reprinted here, by kind permission of the authors, from *An Album of Fluid Motion*, assembled by Milton Van Dyke and published by Parabolic Press in 1982.

I am indebted to Janice D'Amico for her skilful typing of the manuscript, while suffering cheerfully through innumerable revisions. I also thank Changqing (Peter) Hu for drawing the figures from my rough sketches. I am equally indebted to Karl-Friedrich Koch, of the Springer book production department, for his friendly cooperation. Finally, I gratefully acknowledge the continuous support from the National Science Foundation and the Office of Naval Research.

*Constantine M. Dafermos*

# Table of Contents

# Chapter I. Balance Laws

The ambient space for the balance law will be $\mathbb{R}^k$, with typical point $X$. In the applications to Continuum Physics, $\mathbb{R}^k$ will stand for physical space, of dimension one, two or three, in the context of statics; and for space-time, of dimension two, three or four, in the context of dynamics.

The generic balance law in a domain of $\mathbb{R}^k$ will be introduced through its primal formulation, as a postulate that the production of an extensive quantity in any subdomain is balanced by a flux through the boundary; it will then be reduced to a field equation. It is this reduction that renders Continuum Physics mathematically tractable. It will be shown that the divergence form of the field equation is preserved under change of coordinates.

The field equation for the general balance law will be combined with constitutive equations, relating the flux and production density with a state vector, to yield a quasilinear first order system of partial differential equations in divergence form.

It will be shown that symmetrizable systems of balance laws are endowed with companion balance laws which are automatically satisfied by smooth solutions, though not necessarily by weak solutions. The issue of admissibility of weak solutions will be raised.

Solutions will be considered with shock fronts or weak fronts, in which the state vector field or its derivatives experience jump discontinuities across a manifold of codimension one.

The theory of $BV$ functions, which provide the natural setting for solutions with shock fronts, will be surveyed and the geometric structure of BV solutions will be described.

Highly oscillatory weak solutions will be constructed, and a first indication of the stabilizing role of admissibility conditions will be presented.

The setting being Euclidean space, it will be expedient to employ matrix notation, which may be deficient in elegance but is efficient for calculation. The symbol $\mathscr{M}^{r,s}$ will generally denote the space of $r \times s$ matrices and $\mathbb{R}^r$ will be identified with $\mathscr{M}^{r,1}$. Certain objects that are naturally rank (0,2) tensors shall be here represented by matrices. Consequently, standard conventions notwithstanding, in order to retain consistency with matrix operations, gradients must be realized as row vectors and the divergence operator will be acting on row vectors. The unit sphere in $\mathbb{R}^r$ will be denoted throughout by $\mathscr{S}^{r-1}$.

## 1.1 Formulation of the Balance Law

A balance law on an open subset $\mathscr{X}$ of $\mathbb{R}^k$ postulates that the *production* of a (generally vector-valued) "extensive" quantity in any bounded measurable subset $\mathscr{D}$ of $\mathscr{X}$ with finite perimeter is balanced by the *flux* of this quantity through the measure-theoretic boundary $\partial\mathscr{D}$ of $\mathscr{D}$. Note that $\partial\mathscr{D}$ is defined as the set of points whose density relative to both $\mathscr{D}$ and $\mathbb{R}^k\backslash\mathscr{D}$ is nonzero; and $\mathscr{D}$ has finite perimeter when $\partial\mathscr{D}$ has finite $(k-1)$-dimensional Hausdorff measure: $\mathscr{H}^{k-1}(\partial\mathscr{D}) < \infty$. With almost all (with respect to $\mathscr{H}^{k-1}$) points $X$ of $\partial\mathscr{D}$ is associated a vector $N(X) \in \mathscr{S}^{k-1}$ which may be naturally interpreted as the measure-theoretic exterior normal to $\mathscr{D}$ at $X$. A Borel subset $\mathscr{C}$ of $\partial\mathscr{D}$, oriented through the exterior normal $N$, constitutes an *oriented surface*. The reader unfamiliar with the above concepts may consult the brief survey in Section 1.7 and the references on geometric measure theory cited in Section 1.10 or may assume, without much loss, that we are dealing here with open bounded subsets of $\mathscr{X}$ whose topological boundary is a Lipschitz $(k-1)$-dimensional manifold.

The production is introduced through a functional $\mathscr{P}$, defined on bounded measurable subsets $\mathscr{D}$ of $\mathscr{X}$ with finite perimeter, taking values in $\mathbb{R}^n$, and satisfying the conditions

(1.1.1) $$\mathscr{P}(\mathscr{D}_1 \cup \mathscr{D}_2) = \mathscr{P}(\mathscr{D}_1) + \mathscr{P}(\mathscr{D}_2)\ , \quad \text{if } \mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset\ ,$$

(1.1.2) $$|\mathscr{P}(\mathscr{D})| \leq c|\mathscr{D}|\ ,$$

for some constant $c \geq 0$, where $|\mathscr{D}|$ denotes the Lebesgue measure of $\mathscr{D}$.

The flux through $\partial\mathscr{D}$ is induced by a functional $\mathscr{Q}$, defined on the set of oriented surfaces $\mathscr{C}$, which takes values in $\mathbb{R}^n$, and satisfies the conditions

(1.1.3) $$|\mathscr{Q}(\mathscr{C})| \leq c\mathscr{H}^{k-1}(\mathscr{C})\ ,$$

for some constant $c \geq 0$, and

(1.1.4) $$\mathscr{Q}(\mathscr{C}_1 \cup \mathscr{C}_2) = \mathscr{Q}(\mathscr{C}_1) + \mathscr{Q}(\mathscr{C}_2)\ ,$$

for all disjoint Borel subsets $\mathscr{C}_1$, $\mathscr{C}_2$ of $\partial\mathscr{D}$.

Consequently, the balance law states

(1.1.5) $$\mathscr{Q}(\partial\mathscr{D}) = \mathscr{P}(\mathscr{D})$$

for any bounded measurable subset $\mathscr{D}$ of $\mathscr{X}$ with finite perimeter.

## 1.2 Reduction to Field Equations

Due to (1.1.1) and (1.1.2), there is a *production density* $P \in L^\infty(\mathscr{X};\mathbb{R}^n)$ such that

(1.2.1) $$\mathscr{P}(\mathscr{D}) = \int_{\mathscr{D}} P(X)dX\ .$$

Similarly, by virtue of (1.1.3) and (1.1.4), with any bounded measurable subset $\mathscr{D}$ of $\mathscr{X}$, with finite perimenter, is associated a bounded Borel *flux density* function $Q_{\partial\mathscr{D}} : \partial\mathscr{D} \to \mathbb{R}^n$ such that

$$\mathscr{Q}(\mathscr{E}) = \int_{\mathscr{E}} Q_{\partial\mathscr{D}}(X) d\mathscr{H}^{k-1}(X) \tag{1.2.2}$$

holds for any oriented surface $\mathscr{E} \subset \partial\mathscr{D}$. Clearly, if $\mathscr{E} \subset \partial\mathscr{D}_1$ and $\mathscr{E} \subset \partial\mathscr{D}_2$, then $Q_{\partial\mathscr{D}_1}$ and $Q_{\partial\mathscr{D}_2}$ restricted to $\mathscr{E}$ must coincide, a.e. with respect to $\mathscr{H}^{k-1}$.

It is remarkable that the seemingly mild assumptions (1.1.3) and (1.1.4) in conjunction with (1.1.5) imply severe restrictions on the density flux function:

**Theorem 1.2:1** *Under the assumptions* (1.1.3), (1.1.4), (1.1.5), (1.2.1), *and* (1.2.2), *the value of* $Q_{\partial\mathscr{D}}$ *at* $X \in \partial\mathscr{D}$ *depends on* $\partial\mathscr{D}$ *solely through the exterior normal* $N(X)$ *to* $\mathscr{D}$ *at* $X$, *namely, there is a bounded measurable function* $Q : \mathscr{X} \times \mathscr{S}^{k-1} \to \mathbb{R}^n$ *such that*

$$Q_{\partial\mathscr{D}}(X) = Q(X, N(X)) \ , \quad \text{a.e. on } \partial\mathscr{D} \ , \ \text{with respect to } \mathscr{H}^{k-1} \ . \tag{1.2.3}$$

*Furthermore,* $Q$ *depends "linearly" on* $N$, *i.e., there is a flux density field* $A \in L^\infty(\mathscr{X}; \mathscr{M}^{n,k})$ *such that*

$$Q(X, N) = A(X)N \ , \quad \text{a.e. on } \mathscr{X} \ , \tag{1.2.4}$$

*and*

$$\text{div} A = P \ , \tag{1.2.5}$$

*in the sense of distributions.*

**Proof.** To establish (1.2.3), fix $\overline{X} \in \mathscr{X}, \overline{N} \in \mathscr{S}^{k-1}$ and consider any two bounded measurable subsets $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\mathscr{X}$, with finite perimeter, such that $\overline{X} \in \partial\mathscr{D}_1, \overline{X} \in \partial\mathscr{D}_2$, and $N_{\mathscr{D}_1}(\overline{X}) = N_{\mathscr{D}_2}(\overline{X}) = \overline{N}$; see Fig. 1.2.1. The aim is to show that $Q_{\partial\mathscr{D}_1}(\overline{X}) = Q_{\partial\mathscr{D}_2}(\overline{X})$. Let $\mathscr{B}_r$ denote the ball in $\mathbb{R}^k$ of (small) radius $r$ centered at $\overline{X}$. We write the balance law (1.1.5), first for $\mathscr{D} = \mathscr{D}_1 \cap \mathscr{B}_r$ then for $\mathscr{D} = \mathscr{D}_2 \cap \mathscr{B}_r$ and subtract the resulting equations to get

$$\begin{aligned} &\mathscr{Q}(\mathscr{B}_r \cap \partial\mathscr{D}_1) - \mathscr{Q}(\mathscr{B}_r \cap \partial\mathscr{D}_2) \\ &\quad = \mathscr{P}((\mathscr{D}_1 \backslash \mathscr{D}_2) \cap \mathscr{B}_r) - \mathscr{P}((\mathscr{D}_2 \backslash \mathscr{D}_1) \cap \mathscr{B}_r) \\ &\quad\quad - \mathscr{Q}((\mathscr{D}_1 \backslash \mathscr{D}_2) \cap \partial\mathscr{B}_r) + \mathscr{Q}((\mathscr{D}_2 \backslash \mathscr{D}_1) \cap \partial\mathscr{B}_r) \ . \end{aligned} \tag{1.2.6}$$

As $r \downarrow 0$, the first two terms on the right-hand side of (1.2.6) are $O(r^k)$, by virtue of (1.1.2); the last two terms are $o(r^{k-1})$, except possibly on a set of $r$ for which the origin is a point of rarefaction, on account of (1.1.3), since $\mathscr{D}_1$ and $\mathscr{D}_2$ are tangential to each other at $\overline{X}$. Consequently, (1.2.2) and (1.2.6) yield

$$\int_{B_r \cap \partial\mathscr{D}_1} Q_{\partial\mathscr{D}_1}(X) d\mathscr{H}^{k-1}(X) - \int_{B_r \cap \partial\mathscr{D}_2} Q_{\partial\mathscr{D}_2}(X) d\mathscr{H}^{k-1}(X) = o(r^{k-1}) \ . \tag{1.2.7}$$

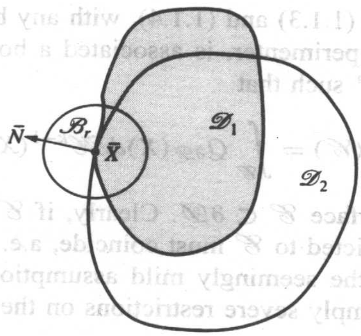


**Fig. 1.2.1.**

Thus, if $\overline{X}$ is a Lebesgue point of both $Q_{\partial\mathscr{D}_1}$ and $Q_{\partial\mathscr{D}_2}$ then $Q_{\partial\mathscr{D}_1}(\overline{X}) = Q_{\partial\mathscr{D}_2}(\overline{X})$.

The proof of (1.2.4) will be attained by means of the celebrated *Cauchy tetrahedron argument*. Consider the standard orthonormal basis $\{E_\alpha : \alpha = 1, \cdots, k\}$ in $\mathbb{R}^k$.

For fixed $\alpha$ and $\overline{X}$, let us apply the balance law to the rectangle $\mathscr{D} = \{X : -\delta < X_\alpha - \overline{X}_\alpha < \varepsilon, |X_\beta - \overline{X}_\beta| < r, \beta \neq \alpha\}$ with $\delta$, $\varepsilon$ and $r$ positive small. Letting $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, one easily deduces $Q(X, -E_\alpha) = -Q(X, E_\alpha)$, a.e. on $\mathscr{X}$.

Now fix $N \in \mathscr{S}^{k-1}$ with nonzero components $N_\alpha$, $\alpha = 1, \cdots, k$, and $\overline{X} \in \mathscr{X}$ which is a Lebesgue point of $Q(\cdot, N)$ as well as of $Q(\cdot, \pm E_\alpha)$, $\alpha = 1, \cdots, k$. Consider the simplex $\mathscr{D} = \{X : (X_\alpha - \overline{X}_\alpha)N_\alpha > -r, \alpha = 1, \cdots, k, (X - \overline{X}) \cdot N < r\}$ with $r$ positive and small. Notice that $\partial\mathscr{D}$ contains a face $\mathscr{C}$ with exterior normal $N$ and faces $\mathscr{C}_\alpha$, $\alpha = 1, \cdots, k$, with exterior normal $-(\operatorname{sgn} N_\alpha)E_\alpha$. Moreover, we have $\mathscr{H}^{k-1}(\mathscr{C}_\alpha) = |N_\alpha|\mathscr{H}^{k-1}(\mathscr{C})$, $\alpha = 1, \cdots, k$. Applying the balance law to this $\mathscr{D}$, dividing through by $\mathscr{H}^{k-1}(\mathscr{C})$ and letting $r \downarrow 0$ yields

$$Q(\overline{X}, N) = \sum_{\alpha=1}^{k} Q(\overline{X}, E_\alpha)N_\alpha\,, \tag{1.2.8}$$

which establishes (1.2.4).

It remains to show (1.2.5). When $A$ is Lipschitz, the balance law takes the form

$$\int_{\partial\mathscr{D}} A(X)N(X)d\mathscr{H}^{k-1}(X) = \int_{\mathscr{D}} P(X)dX \tag{1.2.9}$$

so that (1.2.5) follows directly from Green's theorem. In the general case, when $A$ is merely in $L^\infty$, even though (1.2.9) may no longer make sense for arbitrary $\mathscr{D}$, it will still hold for translates $\mathscr{D}_Y = \{X \in \mathbb{R}^k : X - Y \in \mathscr{D}\}$ of any fixed hypercube $\mathscr{D}$ by almost all $Y$ in a ball $\{Y \in \mathbb{R}^k : |Y| < \varepsilon\}$, with $\varepsilon$ sufficiently small to retain $\overline{\mathscr{D}_Y} \subset \mathscr{X}$. Accordingly, we fix any test function $\psi \in C_0^\infty(\mathbb{R}^k)$ with total mass 1, supported in the unit ball, we rescale it by $\varepsilon$,

(1.2.10)
$$\psi_\varepsilon(X) = \varepsilon^{-k}\psi(\varepsilon^{-1}X) \ ,$$

and use it to mollify, in the customary fashion, the fields $P$ and $A$ on the set $\mathscr{X}_\varepsilon \subset \mathscr{X}$ of points whose distance from $\mathscr{X}^c$ exceeds $\varepsilon$:

(1.2.11)
$$P_\varepsilon = \psi_\varepsilon * P \ , \quad A_\varepsilon = \psi_\varepsilon * A \ .$$

For any hypercube $\mathscr{D} \subset \mathscr{X}_\varepsilon$, we apply Green's theorem to the smooth field $A_\varepsilon$ and use Fubini's theorem to get

(1.2.12)
$$\begin{aligned}
\int_{\mathscr{D}} \mathrm{div} A_\varepsilon(X) dX &= \int_{\partial\mathscr{D}} A_\varepsilon(X)N(X)d\mathscr{H}^{k-1}(X) \\
&= \int_{\partial\mathscr{D}}\int_{\mathbb{R}^k} \psi_\varepsilon(Y)A(X-Y)N(X)dYd\mathscr{H}^{k-1}(X) \\
&= \int_{\mathbb{R}^k} \psi_\varepsilon(Y)\int_{\partial\mathscr{D}_Y} A(Z)N(Z)d\mathscr{H}^{k-1}(Z)dY \\
&= \int_{\mathbb{R}^k} \psi_\varepsilon(Y)\int_{\mathscr{D}_Y} P(Z)dZdY \\
&= \int_{\mathscr{D}}\int_{\mathbb{R}^k} \psi_\varepsilon(Y)P(X-Y)dYdX \\
&= \int_{\mathscr{D}} P_\varepsilon(X)dX \ ,
\end{aligned}$$

whence we conclude $\mathrm{div} A_\varepsilon = P_\varepsilon$ on $\mathscr{X}_\varepsilon$. Letting $\varepsilon \downarrow 0$, yields (1.2.5) on $\mathscr{X}$, in the sense of distributions. This completes the proof.

Conversely, a field equation (1.2.5), with $A \in L^\infty(\mathscr{X};\mathscr{M}^{n,k})$ and $P \in L^\infty(\mathscr{X};\mathbb{R}^n)$, induces a balance law (1.1.5), where $\mathscr{P}$ is defined by (1.2.1), and $\mathscr{Q}$ is obtained from (1.2.2), for some function $Q_{\partial\mathscr{D}} \in L^\infty(\partial\mathscr{D};\mathbb{R}^n)$ identified through its action on test functions $\phi \in C^\infty(\mathbb{R}^k)$:

(1.2.13)
$$\int_{\partial\mathscr{D}} \phi(X)Q_{\partial\mathscr{D}}(X)d\mathscr{H}^{k-1}(X) = \int_{\mathscr{D}} \phi(X)P(X)dX + \int_{\mathscr{D}} A(X)(\mathrm{grad}\,\phi)^T(X)dX \ .$$

Clearly, (1.2.13) is derived formally upon multiplying (1.2.5) by $\phi$, integrating over $\mathscr{D}$ and applying Green's theorem.

In fact, the function $Q_{\partial\mathscr{D}}$ may be constructed, through (1.2.13), even in the more general case where $A \in L^\infty(\mathscr{X};\mathscr{M}^{n,k})$ satisfies a field equation (1.2.5) with $P$ a measure on $\mathscr{X}$. Of course in that case it is no longer generally true that the value of $Q_{\partial\mathscr{D}}$ at $X \in \partial\mathscr{D}$ depends on $\partial\mathscr{D}$ solely through the exterior normal $N(X)$ to $\partial\mathscr{D}$ at $X$. Details may be found in the references cited in Section 1.10.

## 1.3 Change of Coordinates

The divergence form of the field equations of balance laws is preserved under coordinate changes, so long as the fields transform according to appropriate rules.

**Theorem 1.3.1** *Let* $\mathscr{X}$ *be an open subset of* $\mathbb{R}^k$ *and assume that functions* $A \in L^1_{loc}(\mathscr{X}; \mathscr{M}^{n,k})$ and $P \in L^1_{loc}(\mathscr{X}; \mathbb{R}^n)$ *satisfy the field equation*

$$\text{div} A = P \;, \tag{1.3.1}$$

*in the sense of distributions on* $\mathscr{X}$. *Consider any bilipschitz homeomorphism* $X^*$ *of* $\mathscr{X}$ *to a subset* $\mathscr{X}^*$ *of* $\mathbb{R}^k$, *with Jacobian matrix*

$$J = \frac{\partial X^*}{\partial X} \tag{1.3.2}$$

*such that*

$$\det J \geq a > 0 \;, \quad a.e.\ on\ \mathscr{X} \;. \tag{1.3.3}$$

*Then* $A^* \in L^1_{loc}(\mathscr{X}^*; \mathscr{M}^{n,k})$, $P^* \in L^1_{loc}(\mathscr{X}^*; \mathbb{R}^n)$ *defined by*

$$A^* \circ X^* = (\det J)^{-1} A J^T \;, \quad P^* \circ X^* = (\det J)^{-1} P \tag{1.3.4}$$

*satisfy the field equation*

$$\text{div} A^* = P^* \;, \tag{1.3.5}$$

*in the sense of distributions on* $\mathscr{X}^*$.

**Proof.** From (1.3.1) it follows that

$$\int_{\mathscr{X}} [A(\text{grad}\,\phi)^T + P\phi] dX = 0 \tag{1.3.6}$$

holds for any test function $\phi \in C_0^\infty(\mathscr{X})$ and thereby, by completion in $W^{1,\infty}$, for any Lipschitz function $\phi$ with compact support in $\mathscr{X}$.

Given any test function $\phi^* \in C_0^\infty(\mathscr{X}^*)$, consider the Lipschitz function $\phi = \phi^* \circ X^*$, with compact support in $\mathscr{X}$. Notice that $\text{grad}\,\phi = (\text{grad}\,\phi^*)J$. Furthermore, $dX^* = (\det J)dX$. By virtue of these and (1.3.4), (1.3.6) yields

$$\int_{\mathscr{X}^*} [A^*(\text{grad}\,\phi^*)^T + P^*\phi^*] dX^* = 0 \;, \tag{1.3.7}$$

which establishes (1.3.5). The proof is complete.