Workshop on

FINITE TEMPERATURE QCD AND QUARK - GLUON TRANSPORT THEORY

Edited by Liu Lianshou, Wang Enke and Zhang Xiaofei

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PREFACE

Spurred by the experimental project of creating quark-gluon plasma in ultra-relativistic heavy ion collisions at the 'Relativistic Heavy Ion Collider'(RHIC) and 'Large Hadron Collider' (LHC), being in construction at BNL and CERN respectively, substantive progress has been made in recent years in the study of quark-gluon transport theory and high temperature QCD. With the aim of introducing the recent developments in this field to Chinese scholars, the Workshop on Finite Temperature QCD and Quark-Gluon Transport Theory was held in April 1994 at the Institute of Particle Physics, Hua-Zhong Normal University, Wuhan, China.

In this workshop, physicists from the United States, Germany, France, Japan, and China gave lectures on the latest active topics in finite temperature QCD and heavy ion collisions.

U. Heinz who is a pioneer in the study of transport theory for quarks and gluons talked about the framework which allows for theoretical description of the kinetics of quark-gluon plasma. He also reported on new insight into the formation mechanism of strange particles during ultra-relativistic heavy ion collisions and the interesting new details of the strangeness phase diagram.

R. D. Pisarski gave lectures on the possible appearance of disoriented chiral condensates (DCC) from the "quenched" heavy ion collisions and how the position and width of the effective mass of vector mesons are affected by temperature using the theory of chiral symmetry.

Xin-Nian Wang reviewed on purterbative QCD-inspired model for multiple parton production and parton equilibration in heavy ion collisions. He also discussed the effective treatment of soft interactions for which purterbative QCD can not be applied.

Y. Fujimoto discussed the temperature dependent ultraviolet divergences and infrared divergences in massless QED and QCD using dimensional regularization.

We are grateful to all the speakers for their nice lectures and well-prepared manuscripts for the proceedings. Special thanks are extended to Uli Heinz for his encouragement and great help in organizing this workshop.

CONTENTS

QUARK-GLUON TRANSPORT THEORY

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1. Introduction

For a period of nearly ten years now there has been an active experimental programme at CERN and BNL for the study of ultra-relativistic nuclear collisions. The goal of these experiments is to create very hot and dense nuclear matter in the laboratory, and ultimately to achieve conditions under which the matter in the collision zone makes a phase transition to a quark-gluon plasma (QGP). The existence of such a phase at sufficiently high energy density is predicted by quantum chromodynamics and is a consequence of asymptotic freedom.

The quark-gluon plasma is defined to be a state in which quarks and gluons are deconfined, *i.e.* can move around freely over large distances, and are in a state of local thermal equilibrium. This implies that their momentum distributions are of the Fermi or Bose type and thus can be described by only three thermodynamic parameters: the temperature T and two chemical potentials μ_B and μ_s (corresponding to the two conserved quantum numbers of the strong interaction, baryon number and strangeness).

That quarks and gluons are the relevant degrees of freedom at high energies has been amply demonstrated in high energy e^+e^-, pp, and $p\bar{p}$ collisions. What is different in nuclear collisions is that the volume and density of the system may be large enough for the quarks and gluons to rescatter many times and thereby equilibrate, thus exhibiting collective features like color deconfinement, long-range color conductivity, Debye screening and plasma oscillations. These are the specific features we are hoping for in nuclear collisions, and I believe that many of us would be disappointed if we could only prove the existence of quarks and gluons (rather than hadrons) in the early stages of the collision and not convince ourselves in addition that they undergo significant rescattering and a large degree of thermalization, resulting in collective quark-gluon dynamics.

In these lectures I will describe a framework which allows for a theoretical description of the kinetics of a quark-gluon system and its approach to local thermodynamic equilibrium. This formalism aims at the dynamics of the very first, pre-equilibrium stage in relativistic nuclear collisions. Their later evolution, including hydrodynamic expansion and hadronization, will not be discussed here in detail, but have been described at other occasions[1–3]. Here I will at the beginning only give a short summary of the various stages of a heavy-ion collision, in order to establish the general setting.

2. Dynamical Stages of a Nuclear Collision

High energy nuclear collisions proceed through a sequence of dynamical stages each of which leaves certain experimental traces. In this section I present a short overview before proceeding in the following section to a detailed discussion of the quark-gluon kinetics in the early pre-equilibrium stage.

2.1. Primary Nucleon-Nucleon Collisions

The formation of the dense and hot interaction region is initiated by a sequence of high energy collisions between the projectile and target nucleons. The nucleon-nucleon cross section can be separated into two contributions: (a) A "soft" component which produces secondaries with small to moderate (less than, say, 2 GeV) transverse momenta; it is of non-perturbative character and cannot be reliably calculated, but creates most of the multiplicity at c.m. energies below a few hundred GeV. (b) A "hard" component which produces high-$p_\perp$ secondaries and can be calculated with the methods of perturbative QCD; it becomes increasingly important at higher c.m. energies and begins to dominate the production of secondaries at collider energies (above, say, $\sqrt{s} = 0.5$ TeV). It is responsible for the production of jets and minijets, hard direct photons, Drell-Yan dileptons, and of charmed particles, and dominates the spectra of these particles at large invariant masses and transverse momenta (above a few GeV).

2.2. Parton Rescattering, Thermal and Chemical Equilibration

At very high energies the penetrating nuclei can be considered as a stream of partons with momentum distributions given by the nuclear structure functions. Secondaries at central rapidity arise mostly from collisions between low-x partons from these nuclei. Since at low x the parton structure is dominated by gluons and the (considerably smaller) $q\bar{q}$ sea, while the valence quarks have larger x and thus populate the forward and backward nuclear fragmentation regions, the secondaries at central rapidity are initially mostly gluons. At very low x the gluon density in a nucleus becomes very large; thus at high energies, in particular with large nuclei, the density of secondary gluons near central rapidity becomes very high, and they begin to rescatter.

This stage of the collision is described in a kinetic language[4], by solving the equations of motion for the partonic phase-space distribution functions with nuclear structure functions as initial conditions[5]. The numerical implementation[5,6] occurs in the form of parton cascades with perturbative cross sections for parton scattering as well as branching and recombination of off-shell partons. It is worth pointing out a fundamental problem in the kinetic approach to equilibration in nuclear collisions: for an easy description of the quark-gluon plasma phase one would hope for asymptotic freedom to work and to cause a sufficient decrease of the effective strong coupling constant at high temperature, such that the properties of this phase can be described by perturbative methods. This is in fact the underlying spirit in most existing kinetic approaches. On the other hand, if the effective coupling constant were really very small, the particles in the plasma would cease to scatter and would never have a chance to equilibrate. In order to be able to create a quark-gluon plasma (*i.e.* a system of quarks, antiquarks and gluons in thermal equilibrium) on the short time scales characteristic of heavy-ion collisions, we need strong (not weak) interactions. This foils any hopes for a purely perturbative approach to the quark-gluon plasma equation of state and to its dynamics. Although some nonperturbative methods have been developed for the discussion of static properties of a quark-gluon plasma (e.g. its equation of state), much too little is still known about their importance in a dynamical situation. Some interesting recent work on the chaotic dynamics[7] of classical Yang-Mills fields reiterates the necessity for further work in this direction.

These cautionary remarks notwithstanding, intensive recent studies on the basis of the existing numerical codes have shown that the stage of pre-equilibrium dynamics is likely to have important experimental consequences. The semi-hard collisions between the not yet thermalized secondary partons produce quarks and antiquarks (in particular $s\bar{s}$ pairs) from the initially dominating gluons, driving the quark-gluon system towards chemical equilibrium[6,8–10]. They cause substantial additional charm production[8,11,12] and contribute measurably to the spectrum of high-mass dileptons and direct photons[13,14] (depending on the rate of $q\bar{q}$ production from the initial gluons, this source may actually dominate the Drell-Yan rate up the Υ region). They also produce the bulk of the final entropy of the collision region.

Large color octet cross sections and high-gluon densities in collisions between big nuclei at large $\sqrt{s}$ enhance the rate for gluon rescattering, and the gluonic momentum distributions thermalize rapidly[6,8]. For example, for Au-Au collisions at RHIC it is estimated[8,9] that the gluons reach a state of local thermal equilibrium after less than 0.5 fm/c. The cross sections for $q\bar{q}$ (in particular $s\bar{s}$ and $c\bar{c}$) pair production are considerably smaller. Chemical equilibration thus requires more time (about 2–3 fm/c for light quarks and about 3–5 fm/c for strange quarks at RHIC energies[6,9,10]) and may not be completed if the partonic phase does not live long enough before rehadronization. It is thus likely that the partonic rescattering processes in heavy-ion collisions lead to a quark-gluon plasma which is in local thermal, but only partial chemical equilibrium, with the strange sector in particular not being fully saturated.

Still the quoted time scale for strangeness equilibration in the pre-equilibrium parton system is by more than an order of magnitude faster than in a hadronic environment[15]: due to the larger production thresholds in hadronic strangeness producing processes $h_1 + h_2 \rightarrow h_3^s + h_4^{\bar{s}}$ (dictated by the larger constituent rather than the smaller current strange quark mass), strangeness equilibration in, e.g., a hadron gas at temperatures of order 150–200 MeV takes at least a few ten to hundred fm/c. Relative to conventional hadronic rates, strangeness equilibration is thus much faster in a quark-gluon environment. This led to the idea to look for enhanced strangeness production (relative to pp collisions) as an (indirect) signature for an early dense quark-gluon phase[15].

2.3. Hydrodynamic Expansion and Cooling; Hadronization

After local equilibration the further evolution of the dense collision region is governed by the laws of relativistic hydrodynamics. These equations implement (locally) the conservation laws of energy-momentum, baryon number, strangeness, and (in the absence of shocks) entropy. They are formulated in terms of six local parameters, the temperature field $T(x)$, the chemical potentials $\mu_B(x)$ and $\mu_s(x)$, and the hydrodynamic flow velocity $u^\mu(x)$ (with $u^\mu u_\mu = 1$). Their solution needs as input an equation of state which gives the energy density, baryon density and strangeness density as a function of the thermodynamic variables. This equation of state is hard to calculate: although at temperatures far above the deconfinement transition a perturbative QCD approach may be valid, it is very hard to reach such high temperatures experimentally ($T \sim \varepsilon^{1/4}$!), and in practice the system is never in the perturbative regime. Numerical results from lattice QCD, on the other hand, give information only for vanishing baryon density and thus cannot be directly used for nuclear collisions which have nonzero baryon density at least in the nuclear fragmentation regions. Therefore in the hydrodynamic stage one generally makes extensive use of phenomenological equations of state. The most popular choice is a free or perturbatively interacting gas of quarks and gluons subject to an external bag pressure above the deconfinement transition, which is matched below the transition to a hadron resonance gas which includes the measured hadron spectrum with or without mean field interactions or a van-der-Waals-type excluded volume correction.

Due to its internal pressure the plasma expands rapidly against the surrounding vacuum, cooling towards the rehadronization temperature. The expansion rate increases with rising initial temperature and pressure, but the lifetime of the deconfined phase does so, too, albeit more slowly. Hence larger beam energies, leading to higher initial energy densities, prolong the life of the QGP phase. An important contribution to the QGP lifetime comes from the process of rehadronization: since the entropy density in the QGP is much higher than that of a hadron resonance gas (the effective number of degrees of freedom in a high temperature QGP is about an order of magnitude larger than in a hadron gas below $T = 150$ MeV), and the entropy cannot decrease during the expansion, rehadronization requires a large increase in volume, and this takes time. From lattice QCD we know that the drop of the entropy density occurs rather sharply near T_c in the form of a second or weakly first order phase transition[16]. According to the lattice data the largest fraction of the decrease in entropy density happens already in the deconfined phase: this implies that the expanding system spends a large fraction of its lifetime near T_c, as a very slowly cooling non-perturbative quark-gluon phase slightly above T_c which finally hadronizes rapidly at T_c.

Scattering and annihilation of the electrically charged quarks and antiquarks in the QGP lead to thermal radiation of dileptons and photons[17]. These particles interact only electromagnetically and thus escape from the fireball without reinteraction, providing a direct probe of its properties. Fusion of thermal gluons creates more $s\bar{s}$ pairs at a still large rate, driving the system even closer to chemical equilibrium. The hydrodynamic expansion leads to collective flow which adds to the local thermal motion and will leave its imprint on the observed momentum spectra[2,18]. After hadronization, the system may still spend some time in an interacting hadron gas phase before decoupling. This phase can be probed[19] by dileptons from the decay of short-lived vector mesons (mostly the ρ).

2.4. Freeze-out and Decoupling of Hadrons

When the density becomes too low or the expansion rate too fast, local equilibrium can no longer be maintained by collisions among the particles, and the system freezes out. At decoupling the momentum distributions and abundance ratios of the various hadron species are essentially frozen in, the only further modifications being due to the (very important, but calculable) contributions from the decay of unstable resonances.

Nearly all observed particles come from the stage of hadronic freeze-out. Thermal equilibrium wipes out all memory of the earlier stages of the collision, and the observed particle ratios, their y- and $m_\perp$-spectra, and their quantum statistical 2-particle correlations (which can be measured by Hanbury-Brown-Twiss interferometry) provide direct information only about this late stage of the nuclear collision. To extract from hadronic data information about earlier stages always requires some method of extrapolating backwards in time and thus needs a theoretical model. No *direct* proof of QGP formation will ever be possible based on hadronic data alone!

Hadronic freeze-out also provides a background to the direct electromagnetic probes from the earlier stages. Electromagnetic decays of vector mesons (ρ, ω, ϕ) dominate the low-mass dilepton spectrum and thus cover the low-mass part of the thermal QGP radiation. However, the vector meson decays into dileptons are interesting in themselves: The J/ψ traces charm production, and its yield is also sensitive to collective color screening effects in the early quark-gluon phase. The ϕ meson probes enhanced strangeness production. The ratio of the ρ and ω peaks in the dilepton spectrum is sensitive to the lifetime of the hadron gas phase before freeze-out[19]. A really annoying background, however, comes from the electromagnetic π^0 and η decays which completely cover the thermal radiation of direct photons and which must be reconstructed and subtracted with high efficiency before the direct signal can be extracted.

3. Kinetic Theory

Kinetic theory describes the space-time evolution of the phase-space distribution functions which describe the distribution of all the particles of the system in momentum and coordinate space. In classical systems the most important objects of kinetic theory are the (dimensionless) 1-particle distribution functions $f_i(\vec{x},\vec{p},t)$

which give the probability of finding at time t a particle of species i with momentum $\vec{p}$ at point $\vec{x}$. In quantum mechanics, where for a single particle its momentum and coordinate can no longer be simultaneously determined, it is replaced by the Wigner function, the object of quantum kinetic theory. The Wigner function can only be interpreted as a probability when averaged over a sufficiently large phase-space volume to guarantee classical behaviour of the ensemble. However, it still allows to compute macroscopic observables in a very similar way as in the classical case. We will discuss the essential elements of both approaches.

3.1. Classical Kinetic Theory

3.1.1. The Boltzmann-Nordheim-Vlasov kinetic equation

In the absence of collisions, all particles in a classical ensemble move along classical trajectories, and the probability $f(\vec{x},\vec{p},t)$ of finding at time t a particle with momentum $\vec{p}$ at point $\vec{x}$ does not change along these trajectories:

$$\frac{d}{dt}f(\vec{x},\vec{p},t) = \left(\frac{\partial}{\partial t} + \frac{d\vec{x}}{dt}\cdot\vec{\nabla} + \frac{d\vec{p}}{dt}\cdot\vec{\nabla}_p\right) f(\vec{x},\vec{p},t) = 0\,. \tag{1}$$

Inserting the classical equations of motion

$$\frac{d\vec{x}}{dt} = \vec{v}\,, \qquad \frac{d\vec{p}}{dt} = \vec{F} = -\vec{\nabla}U\,, \tag{2}$$

where U is some external or average mean field potential and $\vec{F}$ is the resulting force on the particles, we obtain the *Vlasov equation*

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} - \vec{\nabla}U\cdot\vec{\nabla}_p\right) f(\vec{x},\vec{p},t) = 0\,. \tag{3}$$

The name of Vlasov is connected with the mean-field term $\sim \vec{\nabla}U$ on the left hand side.

Collisions between the particles lead to an additional change of the distribution function with time which can be added to the right hand side of Eq. (3) in the form of a collision term. Since for n-body collisions the collision term involves the n-body distribution function $f(\vec{x}_1,\vec{p}_1,\ldots,\vec{x}_n,\vec{p}_n,t)$, the resulting equation does then no longer close on the single-particle level, and an infinite hierarchy of coupled equations for the n-particle distributions (the BBGKY hierarchy[20]) results. To truncate this hierarchy one can make the so-called *Boltzmann approximation* which neglects all genuine n-body correlations by assuming factorization of the n-particle distribution into a product of 1-particle distributions. Including only 2-body collisions, one thus obtains the Vlasov-Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} - \vec{\nabla}U\cdot\vec{\nabla}_p\right) f(\vec{x},\vec{p},t) = -\tfrac{1}{2}\int \frac{d^3p_1}{(2\pi\hbar)^3}\, d\Omega\, v_{\rm rel}\,\frac{d\sigma}{d\Omega}\,[f\,f_1 - f'\,f_1']\,. \tag{4}$$

The first term describes losses at momentum $\vec{p}$ due to collisions with particles of momentum $\vec{p}_1$ (into final states with momenta $\vec{p}'$ and $\vec{p}'_1$), while the second term describes gains through the inverse scattering process. $v_{\rm rel} = |\vec{v} - \vec{v}_1|$ is the relative velocity of the scattering particles ($\vec{v}_i = \vec{p}_i/m$), and $d\sigma/d\Omega$ is their differential cross section in the center-of-momentum frame. All distribution functions on the right hand side are evaluated at point $(\vec{x}, t)$, e.g. $f_1 \equiv f(\vec{x}, \vec{p}_1, t)$. The factor $\frac{1}{2}$ in front of the collision integral avoids double counting of collisions betwen identical particles.

For indistinguishable particles quantum mechanics creates unavoidable 2-body correlations which are not correctly reproduced by the Boltzmann factorization ansatz. For fermions, for example, the Pauli principle ensures that a kinematically allowed scattering process will not occur if the final momentum state is already occupied. As first discovered by Nordheim[21] and later independently by Uehling ad Uhlenbeck[22], this important quantum effect can be implemented even on the level of classical distribution functions by properly weighting the transition probabilities in the collision integral with final state Pauli suppression factors for fermions and stimulated emission enhancement factors for bosons. The right hand side of Eq. (4) thus has to be replaced by

$$C(\vec{x}, \vec{p}, t) = -\tfrac{1}{2} \int \frac{d^3p_1}{(2\pi\hbar)^3}\, d\Omega\, v_{\rm rel}\, \frac{d\sigma}{d\Omega}\, \left[f f_1 (1 \pm f')(1 \pm f'_1) - f' f'_1 (1 \pm f)(1 \pm f_1) \right] . \tag{5}$$

The kinetic equation with this form of the collision term is known as the Boltzmann-Nordheim-Vlasov equation (sometimes also called Vlasov-Uehling-Uhlenbeck (VUU) equation).

3.1.2. Generalization to relativistic kinematics

For problems with particles moving at relativistic velocities it is useful to bring the kinetic equation into a manifestly covariant form. To this end one writes the 1-particle distribution function as a function of coordinate and momentum 4-vectors x^μ, p^μ, $f(x,p)$. Since classical particles always have to satisfy the energy-momentum relation $E_p = \sqrt{m^2c^4 + \vec{p}^2c^2}$ (which is equivalent to the positive solution of the mass-shell condition $p^2 \equiv p^\mu p_\mu = m^2c^2$), we have to restrict the p-dependence of f onto the mass- shell, by introducing a corresponding δ-function into the integration measure of momentum space:

$$dP = 2\, \theta(p_0)\, \delta(p^2 - m^2c^2) \frac{d^4p}{(2\pi\hbar)^3} = \left. \frac{d^3p}{(2\pi\hbar)^3 p^0} \right|_{p_0 = E/c = \sqrt{\vec{p}^2 + m^2c^2}} . \tag{6}$$

The θ-function ensures that only the positive-energy solutions are selected.

By multiplying with a factor E, Eq. (1) can be rewritten as

$$m \frac{d}{d\tau} f(x,p) = \left[m\dot{x}^\mu \frac{\partial f}{\partial x^\mu} + m\dot{p}^\mu \frac{\partial f}{\partial p^\mu} \right] = 0 , \tag{7}$$

with dots denoting proper time derivatives $d/d\tau$. The coefficients in front of the partial derivatives of f are given by the classical relativistic equations of motion:

$$m\,\dot{x}^{\mu} = m\,u^{\mu} = p^{\mu}\,, \qquad m\,\dot{p}^{\mu} = F^{\mu}\,, \tag{8}$$

where $F^{\mu}(x)$ is the 4-vector of the (external or mean field) force acting on the particles, and the 4-velocity u^{μ} is normalized to 1, $u^{\mu}u_{\mu}/c^2 = 1$, due to the mass-shell constraint.

Generalizing to an ensemble consisting of N different particle species i and adding the collision terms, we obtain the relativistic form of the Boltzmann-Vlasov-Nordheim equation:

$$\left(p_{\mu}\partial^{\mu} + F_{\mu}^{(i)}(x)\partial_p^{\mu}\right) f_i(x,p) = C_i(x,p)\,, \tag{9}$$

with the Boltzmann-Nordheim collision term

$$\begin{aligned} C_i(x,p) = -\tfrac{1}{2}\sum_{j,k,l=1}^{N} \int dP_j\, dP_k\, dP_l\ \big[& f_i f_j (1\pm f_k)(1\pm f_l) W_{ij|kl} \\ & - f_k f_l (1\pm f_i)(1\pm f_j) W_{kl|ij}\big]\,. \end{aligned} \tag{10}$$

The relativistic inelastic transition probabilities are given by

$$W_{ij|kl}(p_i,p_j|p_k,p_l) = (2\pi\hbar)^6\, s\, \sigma_{ij\to kl}(s,\Theta)\, \delta^4(p_i + p_j - p_k - p_l) \tag{11}$$

in terms of the differential inelastic cross section $\sigma_{ij\to kl}(s,\Theta)$ for the process $i(p) + j(p_j) \to k(p_k) + l(p_l)$. This cross section depends on the scattering angle Θ in the center-of-momentum frame,

$$\cos\Theta = \frac{(p^{\mu} - p_j^{\mu})(p_{k\mu} - p_{l\mu})}{(p - p_j)^2}\,, \tag{12}$$

and on the invariant collision energy $s = (p+p_j)^2 = (p_k + p_l)^2$ (s is the square of the energy in the center-of-momentum system). The δ-function in Eq. (11) guarantees energy-momentum conservation.

We see that to write down the left hand side of the Boltzmann-Vlasov equation one always needs the classical equations of motion for the particles in the statistical ensemble. For quarks and gluons carrying color charge and interacting via non-Abelian color fields, we will study them in the following subsection.

3.2. Classical Equations of Motion for Particles with Color and spin

Let us consider the quark color generators $\hat{Q}_a \equiv -t_a = -\frac{\hbar}{2}\lambda_a$ as Heisenberg operators satisfying Heisenberg's equations of motion,

$$\frac{d\hat{Q}_a}{d\tau} = \frac{i}{\hbar}[\mathcal{H}, \hat{Q}_a]\,. \tag{13}$$

If the theory is to be formulated in a relativistically invariant way, the time derivative in Eq. (13) should be with respect to proper time, and the "Hamiltonian" $\mathcal{H}$ (a Lorentz scalar) generating the proper-time evolution is to be taken as Schwinger's "quadratic Dirac Hamiltonian"[23]

$$\mathcal{H} \equiv -\frac{1}{2m}\Big[\Big(i\hbar\partial_\mu + \frac{g}{c}\mathrm{A}_\mu\Big)^2 + \frac{g}{2c}S^{\mu\nu}\mathrm{F}_{\mu\nu} - m^2c^2\Big] . \tag{14}$$

Here $\mathrm{A}_\mu = A^a_\mu t_a$ and $\mathrm{F}_{\mu\nu} = F^a_{\mu\nu}t_a$ are 3×3 matrices in color, with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + (g/c)f_{abc}A^b_\mu A^c_\nu$. $S^{\mu\nu} = (i\hbar/4)[\gamma_\mu, \gamma_\nu]$ is the quark spin tensor, and $(1/2)S^{\mu\nu}\mathrm{F}_{\mu\nu}$, which reduces to $\vec{s}\cdot\vec{\mathrm{B}}$ in the particle's rest frame, is the coupling of the spin to the magnetic field. In the classical c-number limit[a] Eq. (13) leads to the equation of motion[24,25]

$$m\frac{dQ_a}{d\tau} = -\frac{g}{c}f_{abc}\Big(p^\mu A^b_\mu - \frac{1}{mc}p^\alpha \tilde{F}^b_{\alpha\beta}S^\beta\Big)Q^c , \tag{15}$$

where now Q_a are the c-number (*i.e.* commuting) components of an 8-component classical color vector $\vec{Q}$ describing the coupling of a classical colored particle to the eight color potentials A^a_μ. In Eq. (15) $\tilde{\mathrm{F}}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\mathrm{F}^{\mu\nu}$ is the dual field strength tensor, and

$$S^\beta = -\frac{1}{2mc}\epsilon^{\beta\nu\lambda\rho}p_\nu S_{\lambda\rho} \tag{16}$$

relates the classical limit of the spin tensor to a spin 4-vector which is normalized according $p_\mu S^\mu = 0$, $S^\mu S_\mu = -\vec{s}^{\,2}$. Eq. (15) conserves the SU(3) Casimir invariants, *i.e.* the length Q^aQ_a of $\vec{Q}$ and the cubic invariant $d_{abc}Q^aQ^bQ^c$ (d_{abc} are the symmetric structure constants of SU(3)). Thus, the equation describes precession of the classical color vector of the particle due to two effects: direct interaction (color exchange) with an external color field A^a_μ, and coupling of the particle's spin to the color magnetic field. In the absence of particle spin this equation was derived by Wong[26].

If the spin couples to the color magnetic field, it will similarly start to precess, and its equation of motion is given by the c-number limit of $\dot{S}^\mu = \frac{i}{\hbar}[\mathcal{H}, S^\mu]$, namely[24,25]

$$m\frac{dS^\mu}{d\tau} = \frac{g}{c}Q^a\Big[F^{\mu\nu}_a S_\nu + \frac{1}{(mc)^3}(p^\mu S^\nu - p^\nu S^\mu)(D_\nu\tilde{F}_{\alpha\beta})_a p^\alpha S^\beta\Big] . \tag{17}$$

Here

$$(D_\mu)_{ab} = \partial_\mu\delta_{ab} + \frac{g}{c}f_{amb}A^m_\mu \tag{18}$$

is the covariant derivative in the presence of the color field $\mathrm{A}_\mu(x)$. In the absence of inhomogeneities of the external field and of non-Abelian effects, this equation reduces to the BMT equation[27] for a spinning particle with Landé g-factor 2. Had we started from the Yang-Mills Hamiltonian rather than the Dirac Hamiltonian, we

[a] This limit can be considered as the limit $\hbar \to 0$ while the dimension of the representation of the generator simultaneously goes to infinity, such that the expectation value $\langle\hbar\lambda_a/2\rangle$ remains finite.

would have obtained[28] for the spin-1 gluons an analogous equation, but for a g-factor of 1. So with respect to color, we cannot distinguish in the structure of the equations between quarks and gluons: by going to the classical limit (*i.e.* effectively to very high-dimensional representations of the color generators) we lose the difference in color between quarks and gluons. However, in their spin aspects they still remain different: The Landé factor distinguishes quarks from gluons in their coupling to the magnetic field. The spin aspects of the resulting classical kinetic theory have not yet been completely worked out[28], so let us not further elaborate on this point.

Finally, we need an equation of motion for the momentum of the particle. It is given by

$$\begin{aligned} m\frac{dp^\mu}{d\tau} &= \frac{g}{c}Q^a\Big[F_a^{\mu\nu}p_\nu - \frac{1}{mc}(D^\mu\tilde{F}_{\alpha\beta})_a p^\alpha S^\beta\Big] \\ &= \frac{g}{c}Q^a\Big[F_a^{\mu\nu}p_\nu + \frac{1}{2}D^\mu_{ab}(S^{\alpha\beta}F^b_{\alpha\beta})\Big] \,. \end{aligned} \tag{19}$$

In the first term we recognize the (colored version of) the relativistic Lorentz force law. The second term describes the possible gain in energy-momentum due to the space-time variation of the spin magnetic interaction energy in an inhomogeneous color magnetic field.

3.3. Classical Kinetic Equations for the Quark-Gluon Plasma

We can now write down a classical kinetic equation of the Vlasov type for the single-particle phase space distribution for colored and spinning particles. Since the particles' momentum p^μ, the color Q_a, and the spin S_μ all are dynamical variables (*i.e.* evolve in time under the influence of an external or intrinsic mean color field), phase space has to be spanned by the 20 coordinates x^μ, p^μ, Q^a and S^μ. Only in the absence of classical color and spin it reduces to the conventional 8-dimensional phase space (x^μ, p^μ); using for classical particles the mass-shell constraint between p^0 and $\vec{p}$, the latter is further reduced to the 7 well-known dimensions $(\vec{x}, \vec{p}, t)$. In our larger 20-dimensional phase space the integration measure is given by $d\Sigma_\mu\ dP\ dQ\ dS$, where $d\Sigma_\mu$ is the 3-dimensional surface element for some space-like hypersurface Σ, the momentum-space measure is given by Eq. (6), and

$$dQ \sim \delta(Q^aQ_a - q^2)\ \delta(d_{abc}Q^aQ^bQ^c - \tilde{q}^3)\ d^8Q\,, \tag{20}$$

$$dS \sim \delta(p_\mu S^\mu)\,\delta(S^\mu S_\mu + s^2)\ d^4S\,. \tag{21}$$

The various δ-functions fix the normalization constraints for Q^a and S^μ, and the proportionality constants can be conveniently chosen to normalize the measures in color and spin space to unity.

The probability to find a classical particle at a given point in this phase space is given by the 1-particle distribution function $f(x, p, Q, S)$. It has to be a Lorentz scalar and gauge invariant. A gauge and Lorentz invariant expression for the time evolution of $f(x,p,Q,S)$ is given by

$$m\frac{df}{d\tau} \equiv \left[p^\mu \frac{\partial f}{\partial x^\mu} + m\dot{p}^\mu \frac{\partial f}{\partial p^\mu} + m\dot{Q}^a \frac{\partial f}{\partial Q^a} + m\dot{S}^\mu \frac{\partial f}{\partial S^\mu}\right] = C(x,p,Q,S). \tag{22}$$

C on the right hand side is a collision term describing short-range collision processes, while the various terms on the left are easily seen to describe long-range effects due to the mean color fields in the system: Inserting Eqs. (15, 19) and leaving all spin effect aside (to keep the expressions manageable), we obtain the following equation for the 1-particle distribution function of a plasma of classical colored particles[29]:

$$p^\mu\left[\partial_\mu - \frac{g}{c}Q_a F^a_{\mu\nu}(x)\partial^\nu_p - \frac{g}{c}f_{abc}A^b_\mu(x)Q^c\partial^a_Q\right]f(x,p,Q) = C(x,p,Q). \tag{23}$$

If there are antiparticles involved, their distribution function $\bar{f}(x,p,Q)$ obeys a similar equation, with Q^a replaced by $-Q^a$ (*i.e.* the second term changes sign).

These equations are closed by the Yang-Mills equation for the mean color field A_μ

$$\begin{aligned}(D_\mu F^{\mu\nu})_a(x) &= -g j^\nu_a(x) \qquad (24)\\ &= \frac{gc}{\hbar}\int p^\nu Q_a\left[f(x,p,Q) - \bar{f}(x,p,Q) + G(x,p,Q)\right]dPdQ\,.\end{aligned}$$

The color current on the right hand side of the Yang-Mills equation has been expressed in terms of moments of the distribution functions $f, \bar{f}$, and G for the quarks, antiquarks, and gluons, respectively.

Eqs. (23, 24) together form the basis of a relativistic kinetic description for a plasma of colored particles. The mean field terms on the left hand side of Eq. (23) generalize those known from the usual Vlasov equation for electromagnetic plasmas; however, in addition to the drift in momentum induced by the electric and magnetic fields (non-relativistically the combination $\vec{E} + \vec{v}/c\times\vec{B}$ occurs as the coefficient of the momentum derivative of f), there are now also drift terms in the color sector of phase space: since the non-Abelian mean field carries color, interactions between particles and the mean field generally lead to a color transfer.

The collision term on the right hand side of Eq. (23) couples the 1-particle distribution function to two-body correlations. So actually Eqs. (23, 24) generally do not close; closure can be obtained, however, by factorizing the two-body correlations into products of single-particle distribution functions (Boltzmann approximation). Without this approximation further kinetic equations are needed describing the evolution of the 2-body distribution function which then again couples to 3-body correlations, and so on. This BBGKY hierarchy of coupled equations has not been constructed yet for the non-Abelian case; in principle it should emerge from the quantum mechanical formulation of Section 4 in the classical limit, but this has so far not been shown explicitly.

3.4. Color Moment Equations

From the kinetic equations (23) one can construct several infinite hierarchies of moment equations, by forming moments involving powers of the color vector Q^a, of the momentum vector p^μ, or both. The color moment equations prove useful later when comparing with the quantum mechanical formulation, since it turns out that the lowest color moments of $f(x,p,Q)$ can be identified with the classical limit of the color components of the Wigner function. The two lowest moments of the momentum operator formed with these color moments then lead to equations of motion for macroscopic entities, namely the space-time densities of energy-momentum, baryon number and color current, *i.e.* they yield a chromohydrodynamic description of the plasma[25].

To derive the color moment equations, let us define the color singlet, octet, etc. distribution functions as the following moments of $f(x,p,Q)$:

$$
\begin{aligned}
f(x,p) &= \int f(x,p,Q)\, dQ \,, \\
f_a(x,p) &= \int Q_a\, f(x,p,Q)\, dQ \,, \\
f_{ab}(x,p) &= \int Q_a Q_b\, f(x,p,Q)\, dQ \,, \quad \text{etc.}
\end{aligned}
\tag{25}
$$

For these one obtains from Eq. (23), by taking appropriate color moments,

$$
p^\mu \partial_\mu f(x,p) = \frac{g}{c} p^\mu F^a_{\mu\nu}(x)\, \partial^\nu_p f_a(x,p) + \int C(x,p,Q)\, dQ \,, \tag{26}
$$

$$
\begin{aligned}
p^\mu \Big[\partial_\mu \delta_{ac} &+ \frac{g}{c} f_{amc} A^m_\mu(x)\Big]\, f_c(x,p) \\
&= \frac{g}{c} p^\mu F^b_{\mu\nu}(x)\, \partial^\nu_p f_{ab}(x,p) + \int Q_a C(x,p,Q)\, dQ \,,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
p^\mu \Big[\partial_\mu \delta_{ac}\delta_{bd} &+ \frac{g}{c}(\delta_{ac} f_{bmd} + \delta_{bd} f_{amc}) A^m_\mu(x)\Big]\, f_{cd}(x,p) \\
&= \frac{g}{c} p^\mu F^c_{\mu\nu}(x)\, \partial^\nu_p f_{abc}(x,p) + \int Q_a Q_b C(x,p,Q)\, dQ \,, \quad \text{etc.}
\end{aligned}
\tag{28}
$$

Classically, all these moments are independent. Quantum mechanically, the color charges Q_a do not commute, and the color algebra between them allows in the case of quarks (where $Q_a \leftrightarrow -\hbar\lambda_a/2$) to express[25,29] the second color moment f_{ab} in terms of f and f_a:

$$
f_{ab} = \delta_{ab} \frac{\hbar^2}{6} f - \frac{\hbar}{2} d_{abc}\, f_c \,. \tag{29}
$$

Hence for quarks the color hierarchy can be truncated by hand by imposing Eq. (29) even on the classical level and rewriting Eq. (27) as

$$
\begin{aligned}
p^\mu \Big[\delta_{ac}\partial_\mu &+ \frac{g}{c}\, f_{amc} A^m_\mu(x) + \frac{g\hbar}{2c} d_{amc} F^m_{\mu\nu}(x)\, \partial^\nu_p\Big] f_c(x,p) \\
&= \frac{g\hbar^2}{6c}\, p^\mu F^a_{\mu\nu}(x)\, \partial^\nu_p f(x,p) + \int Q_a C(x,p,Q)\, dQ \,.
\end{aligned}
\tag{30}
$$