Graduate Series in Mathematics 4

Zhang Chuanyi

Almost Periodic Type Functions and Ergodicity

(概周期型函数和遍历性)

Science Press
Beijing/New York

Kluwer Academic Publishers
Dordrecht/Boston/London

Almost Periodic Type Functions and Ergodicity

By

Zhang Chuanyi

Department of Mathematics
Harbin Institute of Technology
Harbin, Heilongjiang
The People's Republic of China

Science Press
Beijing/New York

Kluwer Academic Publishers
Dordrecht/Boston/London

Library of Congress. Cataloging-in-Publication Data

ISBN 7-03-010489-7 (Science Press Beijing)
ISBN 1-4020-1158-X (Kluwer Academic Publishers)

Published by Kluwer Academic Publishers,
P. O. Box 17,3300 AA Dordrecht, The Netherlands
and Science Press, Beijing The People's Republic of China

Sold and distributed in North, Central and South America
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U. S. A.

Sold and distributed in the People's Republic of China
by Science Press, Beijing.

In all other countries, sold and distributed
by Kluwer Academic Publishers,
P. O. Box 322,3300 AH Dordrecht, The Netherlands.

Printed on acid-free paper

Printed in Beijing

Preface

The theory of almost periodic functions was first developed by the Danish mathematician H. Bohr during 1925–1926. Then Bohr's work was substantially extended by S. Bochner, H. Weyl, A. Besicovitch, J. Favard, J. von Neumann, V.V. Stepanov, N.N. Bogolyubov, and others. Generalization of the classical theory of almost periodic functions has been taken in several directions. One direction is the broader study of functions of almost periodic type. Related this is the study of ergodicity. It shows that the ergodicity plays an important part in the theories of function spectrum, semigroup of bounded linear operators, and dynamical systems. The purpose of this book is to develop a theory of almost periodic type functions and ergodicity with applications—in particular, to our interest—in the theory of differential equations, functional differential equations and abstract evolution equations. The author selects these topics because there have been many (excellent) books on almost periodic functions and relatively, few books on almost periodic type and ergodicity. The author also wishes to reflect new results in the book during recent years.

The book consists of four chapters. In the first chapter, we present a basic theory of four almost periodic type functions. Section 1.1 is about almost periodic functions. To make the reader easily learn the almost periodicity, we first discuss it in scalar case. After studying a classical theory for this case, we generalize it to finite dimensional vector-valued case, and finally, to Banach-valued (including Hilbert-valued) situation. Section 1.2 is about asymptotically almost periodic functions. Since the reader has some understanding of the almost periodicity for the case of both scalar-valued and vector-valued after studying Section 1.1, we develop the theory of asymptotically almost periodic functions mainly in an abstract vector-valued case. Similar development is applied to the theory of weakly almost periodic functions in Section 1.3. In Section 1.4 we show the approximation theorem for almost periodic functions and the unique decomposition theorem for weakly almost periodic functions. In Section 1.5, we investigate the theory of pseudo almost periodic functions. Until now, we have presented all of the almost periodic type functions we will devote to in the book. In Section 1.6 we apply pseudo almost periodic functions to the converse problems of the Fourier expansion. In the last section of the chapter, Section 1.7, we investigate almost periodic type sequences. This kind of the sequences proved to be useful in differential and difference equations.

In Chapter two, we apply the theory developed in the previous chapter to differential equations. Section 2.1 and Section 2.2 deal with the applications in ordinary and partial differential equations, respectively. In Section 2.3, we first develop a theory of means and introversions and then apply it to some nonlinear differential equations. We discuss the regularity and exponential dichotomy in Section 2.4. In Section 2.5, we deal with equations with piecewise constant argument. In Section 2.6, we solve equations with unbounded inhomogeneous parts. Finally in Section 2.7, we study exponential dichotomy in terms of topological equivalence and structural stability.

Chapter three is about ergodicity and abstract differential equations. We discuss the ergodicity and regularity in the first section. In Section 3.2 we apply the ergodicity to some nonlinear differential equations. For the needs of further applications of almost periodic type functions, we present some basic knowledge of semigroups of bounded linear operators in Section 3.3. Then in Section 3.4 we discuss delay differential equations. In Section 3.5 we develop a theory of function spectrum. In the last section of the chapter, Section 3.6, we solve abstract Cauchy problems and study asymptotical stability of the solutions.

The last chapter of the book is Chapter four. In the chapter, we will apply the theory of ergodicity to averaging methods. For this purpose, we first investigate further properties of an ergodic function in Section 4.1. Then we apply the theory to deal with the quantitative aspect of averaging methods in Section 4.2. In Section 4.3 we present some results on critical theory of some system. Then we deal with the qualitative aspect of averaging methods in Section 4.4. In the last section of the chapter, Section 4.5, we apply the averaging method to functional differential equations and its discrete analogue.

To read the book, we assume the reader has some basic knowledge of Banach space and Hilbert space from Functional Analysis, some knowledge of Ordinary Differential Equations such as existence and unique theorems. Except these, the book is basically self-contained.

The author would like to thank Professor C. Wu for his regards. The author appreciates the supports from Publishing Fund of HIT and NSF of China.

Chuanyi Zhang

Department of Mathematics
Harbin Institute of Technology
Harbin, Heilongjiang
The People's Republic of China

Contents

Chapter 1
Almost periodic type functions

In this chapter we will investigate four function spaces. The four spaces have some common properties. Because of this, they all have similar applications in many areas. In particular, to our interest in this book, they have many applications in the theory of differential equations. At the same time, we will see that they also have substantial differences. We have to characterize them in quite different ways. This also leads them to different applications.

1.1 Almost periodic functions

In this section, we present some classical results on almost periodic functions. First we concentrate numerical almost periodic functions on $\mathbb{R}$. To apply these functions to some differential equations, we then investigate functions almost periodic in $t \in \mathbb{R}$ and uniform on compact subsets of n-dimensional complex space. Finally we briefly introduce Banach value almost periodic functions.

1.1.1 Numerical almost periodic functions

Throughout the book, $\mathbb{R}$ denotes the real line, $\mathbb{C}$ denotes the set of complex numbers, and $\mathcal{C}(\mathbb{R})$ denotes the set of all bounded, complex-valued, continuous functions on $\mathbb{R}$. Define the norm for each $f \in \mathcal{C}(\mathbb{R})$ by $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$, then $\mathcal{C}(\mathbb{R})$ becomes a Banach space. A function space $\mathcal{F}$ is called C^*-algebra if $\mathcal{F}$ is a Banach space and is closed under function multiplication and conjugation. $\mathcal{C}(\mathbb{R})$ is obviously a C^*-algebra.

A trigonometric polynomial S is a function of the form

$$S(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$$

where $\lambda_k \in \mathbb{R}$ and $c_k \in \mathbb{C}$. The functions $ce^{i\lambda t}$ are periodic. However, the sum $c_1 e^{i\lambda_1 t} + c_2 e^{i\lambda_2 t}$ will not be periodic if the ratio of λ_1 to λ_2 is not rational. Thus, a trigonometric polynomial may not be periodic. From the modern point of view, the class of periodic functions on $\mathbb{R}$ is not particularly nice because it does not form a linear space. To overcome this, we seek a class of functions with better structural property and also having properties similar to that of periodic functions. The class we are seeking is now defined in the following

Definition 1.1 *A function $f \in \mathcal{C}(\mathbb{R})$ is said to be almost periodic if for every $\epsilon > 0$ there exists a trigonometric polynomial S_ϵ such that*

$$\|f - S_\epsilon\| < \epsilon.$$

Denote by $\mathcal{AP}(\mathbb{R})$ the set of all such functions.

From the definition above, one sees that $\mathcal{AP}(\mathbb{R})$ is the completion in $\mathcal{C}(\mathbb{R})$ of trigonometric polynomials. It is well known that by Fejér sums, any continuous, periodic function is a uniform limit of trigonometric polynomials. So, all continuous, periodic functions are in $\mathcal{AP}(\mathbb{R})$ (the reader may also easily get the same conclusion by Theorem 1.10 (v) below). Thus, $\mathcal{AP}(\mathbb{R})$ is a quite natural object to study.

Since the set of all trigonometric polynomials is closed under function multiplication and conjugation, so is $\mathcal{AP}(\mathbb{R})$.

For a function f on $\mathbb{R}$, the translate of f by $s \in \mathbb{R}$ is the function $R_s f$ such that $R_s f(t) = f(t + s)$ for all $t \in \mathbb{R}$. We call a function set $\mathcal{F}$ translation invariant if $\{R_s f : f \in \mathcal{F}, s \in \mathbb{R}\} \subset \mathcal{F}$. Note that the set of all trigonometric polynomials is translation invariant and for any f, $g \in \mathcal{C}(\mathbb{R})$,

$$\|R_s f - R_s g\| \leq \|f - g\|.$$

The completion $\mathcal{AP}(\mathbb{R})$ is also translation invariant.

Combining the discussions in the last two paragraphs we have the following theorem on the structural property of $\mathcal{AP}(\mathbb{R})$.

Theorem 1.2 *$\mathcal{AP}(\mathbb{R})$ is a translation invariant, C^*-subalgebra of $\mathcal{C}(\mathbb{R})$ containing the constant functions.*

By Theorem 1.2, if f, $g \in \mathcal{AP}(\mathbb{R})$ then the following functions

$$f \pm g, \quad f \cdot g, \quad \bar{f}, \quad cf, \quad R_s f$$

are all in $\mathcal{AP}(\mathbb{R})$, where c is a constant and $s \in \mathbb{R}$. Is the derivative f' also in $\mathcal{AP}(\mathbb{R})$? The following two corollaries show that uniform continuity characterizes the almost periodicity of f'.

Corollary 1.3 *An $f \in \mathcal{AP}(\mathbb{R})$ is uniformly continuous on $\mathbb{R}$.*

Proof. Note that any function $c_k \exp\{i\lambda_k \cdot\}$ is uniformly continuous on $\mathbb{R}$ and so is the sum of a finite such functions. For $\epsilon > 0$ there is a trigonometric polynomial S_ϵ such that $\|f - S_\epsilon\| < \epsilon/3$. It follows from the uniform continuity of S_ϵ that there exists a $\delta > 0$ such that for any x_1, $x_2 \in \mathbb{R}$, $|x_1 - x_2| < \delta$, one has $|S_\epsilon(x_1) - S_\epsilon(x_2)| < \epsilon/3$. So

$$|f(x_1) - f(x_2)| \leq |f(x_1) - S_\epsilon(x_1)| + |S_\epsilon(x_1) - S_\epsilon(x_2)| + |S_\epsilon(x_2) - f(x_2)| < \epsilon.$$

Thus f is uniformly continuous. □

Corollary 1.4 *Let $f \in \mathcal{AP}(\mathbb{R})$ be such that its derivative function f' is uniformly continuous on $\mathbb{R}$. Then $f' \in \mathcal{AP}(\mathbb{R})$.*

Proof. Let $f = f_1 + if_2$, where f_1 and f_2 are real functions. Consider the functions

$$\varphi_n(t) = n\left[f\left(t + \frac{1}{n}\right) - f(t)\right], \qquad n = 1, 2, \cdots.$$

Since

$$\varphi_n(t) = n\left[f\left(t + \tfrac{1}{n}\right) - f(t)\right] = f_1'\left(t + \tfrac{\theta_n}{n}\right) + if_2'\left(t + \tfrac{\tau_n}{n}\right)$$
$$(0 < \theta_n, \tau_n < 1),$$

the uniform continuity of f' implies that for $\epsilon > 0$ there is an n_0 such that

$$\|\varphi_n - f'\| < \epsilon \qquad (n \geq n_0).$$

The functions φ_n are almost periodic and consequently f' is almost periodic. □

It is well known that any continuous, periodic function f with periodic 2ω associates a Fourier series

$$f(t) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\omega_k t},$$

where

$$\omega_k = \frac{k\pi}{\omega}, \ A_k = \frac{1}{2\omega}\int_0^{2\omega} f(t)e^{-i\omega_k t}dt, \quad k = 0, \pm 1, \pm 2, \cdots.$$

f also satisfies Parseval's equality:

$$\frac{1}{2\omega}\int_0^{2\omega} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |A_k|^2.$$

The function f is completely defined by its Fourier series. More precisely, the following so-called uniqueness theorem is true: Distinct periodic functions have distinct Fourier series.

Next, we show that $\mathcal{AP}(\mathbb{R})$ has a similar theory of Fourier Analysis. In order to introduce the Fourier series of $f \in \mathcal{AP}(\mathbb{R})$, we need first to establish the following result.

Theorem 1.5 *If $f \in \mathcal{AP}(\mathbb{R})$ then the limit*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt$$

exists uniformly with respect to $a \in \mathbb{R}$. Furthermore, the limit is independent of a.

Proof. We first show the theorem in the case that f is a trigonometric polynomial. Let

$$f(t) = S(t) = c_0 + \sum_{k=1}^{n} c_k e^{i\lambda_k t},$$

where $\lambda_k \neq 0$, $k = 1, 2, \cdots, n$. It follows that

$$\frac{1}{2T}\int_{-T+a}^{T+a} S(t)dt = c_0 + \sum_{k=1}^{n} c_k \frac{e^{i\lambda_k(T+a)} - e^{i\lambda_k(-T+a)}}{2i\lambda_k T}.$$

So

$$\left| \frac{1}{2T}\int_{-T+a}^{T+a} S(t)dt - c_0 \right| \leq \frac{1}{T}\sum_{k=1}^{n} \left| \frac{c_k}{\lambda_k} \right|.$$

This implies that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)dt = c_0$$

uniformly with respect to $a \in \mathbb{R}$.

If f is an arbitrary function in $\mathcal{AP}(\mathbb{R})$ then for $\epsilon > 0$ there is a trigonometric polynomial S such that

$$\|f - S\| < \frac{\epsilon}{3}. \tag{1.1}$$

We can find a number T_0 such that when $T_1, T_2 > T_0$, we have

$$\left| \frac{1}{2T_1} \int_{-T_1+a}^{T_1+a} S(t)dt - \frac{1}{2T_2} \int_{-T_2+a}^{T_2+a} S(t)dt \right| < \frac{\epsilon}{3} \qquad (a \in \mathbb{R}). \tag{1.2}$$

It follows from (1.1) and (1.2) that when $T_1, T_2 > T_0$, , we have

$$\begin{aligned} &\left| \frac{1}{2T_1} \int_{-T_1+a}^{T_1+a} f(t)dt - \frac{1}{2T_2} \int_{-T_2+a}^{T_2+a} f(t)dt \right| \\ \leq\ & \frac{1}{2T_1} \int_{-T_1+a}^{T_1+a} |f(t) - S(t)|dt \\ &+ \left| \frac{1}{2T_1} \int_{-T_1+a}^{T_1+a} S(t)dt - \frac{1}{2T_2} \int_{-T_2+a}^{T_2+a} S(t)dt \right| \\ &+ \frac{1}{2T_2} \int_{-T_2+a}^{T_2+a} |f(t) - S(t)|dt < \epsilon. \end{aligned}$$

Let us show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+a}^{T+a} f(t)dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt. \tag{1.3}$$

Thus the limit is independent of a. We may assume, without loss of generality, that $a > 0$. Then

$$\begin{aligned} &\left| \frac{1}{2T} \int_{-T+a}^{T+a} f(t)dt - \frac{1}{2T} \int_{-T}^{T} f(t)dt \right| \\ =\ & \frac{1}{2T} \left| \int_{T}^{T+a} f(t)dt - \int_{-T}^{-T+a} f(t)dt \right| \leq \frac{2a\|f\|}{2T}. \end{aligned}$$

So (1.3) holds. The proof is complete. □

Definition 1.6 *Let $f \in C(\mathbb{R})$. If the limit*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt$$

exists then we call the limit mean of f and denote it by $M(f)$.

In practice, we often write a mean as

$$M(f(\cdot)) \qquad \text{or} \qquad M(f(t)).$$

It follows from Theorem 1.5 that $M(R_s f) = M(f)$ for all $f \in \mathcal{AP}(\mathbb{R})$ and $s \in \mathbb{R}$. That is, M is a translation invariant mean on $\mathcal{AP}(\mathbb{R})$. It is easy to see that M is indeed a positive, bounded linear functional on $\mathcal{AP}(\mathbb{R})$ with $\|M\| = 1$.

If $f \in \mathcal{AP}(\mathbb{R})$, then the mean $M(f)$ can also be calculated by

$$M(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt.$$

We leave the proof to the reader, or refer to Section 3.1.

For $\lambda \in \mathbb{R}$ and $f \in \mathcal{AP}(\mathbb{R})$ since the function $fe^{-i\lambda\cdot}$ is in $\mathcal{AP}(\mathbb{R})$, the mean exists for this function. We write

$$a(\lambda) = M(fe^{-i\lambda\cdot}).$$

Let $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ be in $\mathbb{R}$ and let $c_1,\ c_2,\ \cdots,\ c_n$ be any complex numbers. Consider the following function

$$\varphi(c_1, c_2, \cdots, c_n) = M\left(\left|f - \sum_{k=1}^{n} c_k e^{i\lambda_k\cdot}\right|^2\right).$$

Lemma 1.7 *The function φ assumes a minimum value for $c_k = a(\lambda_k)$, $k = 1,\ 2,\ \cdots,\ n$ and*

$$\sum_{k=1}^{n} |a(\lambda_k)|^2 \le M(|f|^2). \tag{1.4}$$

Proof. An elementary calculation shows that

$$M(e^{i\lambda_k\cdot}\cdot e^{-i\lambda_m\cdot}) = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \ne m. \end{cases}$$

Therefore

$$\begin{aligned} & M\left(\left|f - \sum_{k=1}^{n} c_k e^{i\lambda_k\cdot}\right|^2\right) \\ = \ & M(|f|^2) - \sum_{k=1}^{n} \bar{c}_k M(fe^{-i\lambda_k\cdot}) - \sum_{k=1}^{n} c_k M(\bar{f}e^{i\lambda_k\cdot}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\sum_{m=1}^{n} c_k\bar{c}_m M(e^{i\lambda_k\cdot}\cdot e^{-i\lambda_m\cdot}) \\
=\;& M(|f|^2)-\sum_{k=1}^{n}\bar{c}_k a(\lambda_k)-\sum_{k=1}^{n}c_k\bar{a}(\lambda_k)+\sum_{k=1}^{n}|c_k|^2 \\
=\;& M(|f|^2)+\sum_{k=1}^{n}|c_k-a(\lambda_k)|^2-\sum_{k=1}^{n}|a(\lambda_k)|^2.
\end{aligned}
$$

Consequently, the function φ takes the minimum value for $c_k = a(\lambda_k)$ and the value is $M(|f|^2) - \sum_{k=1}^{n} |a(\lambda_k)|^2$. One has (1.4) because $\varphi \geq 0$. The proof is complete. □

The following two theorems are of special important in the theory of almost periodic functions. Because of them, Fourier Analysis can be carried out on $\mathcal{AP}(\mathbb{R})$. The proofs all depend on Lemma 1.7.

Theorem 1.8 *Let $f \in \mathcal{AP}(\mathbb{R})$. Then there exists at most a countable set of λ's for which $a(\lambda) \neq 0$.*

Proof. By (1.4) it follows that there can exist only a finite number of $\lambda's$ for which $|a(\lambda)| \geq 1$. Similarly one shows that for any n there exists only a finite numbers of $\lambda's$ for which

$$\frac{1}{n+1} \leq |a(\lambda)| < \frac{1}{n}.$$

Hence, the set of $\lambda's$ for which $a(\lambda) \neq 0$ is a countable union of finite sets, and consequently it is countable. □

For a function $f \in \mathcal{AP}(\mathbb{R})$, the set

$$\mathrm{Freq}(f) = \{\lambda \in \mathbb{R} : \; a(\lambda) \neq 0\}$$

is called the frequency set of f. Members of Freq(f) are called the Fourier exponents of f and $a(\lambda)$'s are called the Fourier coefficients of f. By theorem 1.8, Freq(f) is countable. Let $\mathrm{Freq}(f) = \{\lambda_k\}$ and $A_k = a(\lambda_k)$. Thus f can associate a Fourier series:

$$f(t) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k t}.$$

Theorem 1.9 *If $f \in \mathcal{AP}(\mathbb{R})$ then Parseval's equality*

$$\sum_{k=1}^{\infty} |A_k|^2 = M(|f|^2) \tag{1.5}$$

holds.

Proof. By inequality (1.4) for any n one has

$$\sum_{k=1}^{n} |A_k|^2 \leq M(|f|^2).$$

This shows that

$$\sum_{k=1}^{\infty} |A_k|^2 \leq M(|f|^2).$$

Let S be any trigonometric polynomial. Set $S^* = 0$ if none of the exponents of f occurs among the Fourier exponents of S, and $S^*(t) = \sum A_k e^{i\lambda_k t}$, the summation being extended over those $k's$ for which λ_k is a Fourier exponent common to the functions f and S.

Since f is almost periodic, there is a sequence of trigonometric polynomials $\{S_n\}$ such that

$$\|f - S_n\| < \frac{1}{\sqrt{n}}.$$

Then we have

$$M(|f - S_n|^2) \leq \frac{1}{n}.$$

Applying Lemma 1.7 we have

$$M(|f - S_n^*|^2) \leq M(|f - S_n|^2) \leq \frac{1}{n}.$$

But

$$M(|f - S_n^*|^2) = M(|f|^2) - \sum_k |A_k|^2,$$

where the sum is extended over those $k's$ for which λ_k is a Fourier exponent of S_n. Hence

$$M(|f|^2) \leq \sum_k |A_k|^2 + \frac{1}{n},$$

with the same conventions regarding the summation. Furthermore, we shall have that

$$M(|f|^2) \leq \sum_{k=1}^{\infty} |A_k|^2 + \frac{1}{n},$$

and since n is arbitrarily large, we obtain

$$M(|f|^2) \leq \sum_{k=1}^{\infty} |A_k|^2.$$

The proof is complete. □