NUMERICAL RECIPES in C++

The Art of Scientific Computing

Second Edition

William H. Press
Saul A. Teukolsky
William T. Vetterling
Brian P. Flannery

Numerical Recipes in C++

The Art of Scientific Computing

Second Edition

William H. Press

Los Alamos National Laboratory

Saul A. Teukolsky

Department of Physics, Cornell University

William T. Vetterling

Polaroid Corporation

Brian P. Flannery

EXXON Research and Engineering Company

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC, 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

Some sections of this book were originally published, in different form, in *Computers in Physics* magazine, © American Institute of Physics, 1988–1992.

First Edition originally published 1988; Second Edition originally published 1992; C++ edition originally published 2002.

This printing is corrected to software version 2.10

Printed in the United States of America
Typeface Times 10/12 pt. *System* TEX [AU]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Numerical recipes in C++ : the art of scientific computing / William H. Press
... [et al.]. – 2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-75033-4
1. C++ (Computer program language) 2. Numerical analysis. I. Press, William H.
QA76.73.C153 N85 2002
519.4'0285'5133–dc21 2001052699

ISBN 0 521 75033 4 Hardback
ISBN 0 521 75034 2 Example book in C++
ISBN 0 521 75037 7 C++/C CD-ROM (Windows/Macintosh)
ISBN 0 521 75035 0 Complete CD-ROM (Windows/Macintosh)
ISBN 0 521 75036 9 Complete CD-ROM (UNIX/Linux)

Preface to the C++ Edition

C++ has gradually become the dominant language for computer programming, displacing C and Fortran even in many scientific and engineering applications. This version of *Numerical Recipes* contains the entire text of the Second Edition with all the programs presented in C++.

C++ poses special problems for numerical work. In particular, it is difficult to treat vectors and matrices in a manner that is simultaneously efficient and yet allows programming with high-level constructs. The fact that there is still no universally accepted standard library for doing this makes the problem even more difficult for authors of a book like this one. In Chapter 1 and the Appendices we describe how we have solved this problem. The default option is for you, the reader, to use a very simple class library that we provide. You can be up and running in a few minutes. We also show you how you can alternatively use any other matrix/vector class library of your choosing. This may take you a few minutes to set up the first time, but thereafter will provide transparent access to the Recipes with essentially no loss in efficiency.

We have taken this opportunity to respond to a clear consensus from our C readers, and converted all arrays and matrices to be "zero-based." We have also taken this opportunity to fix errors in the text and programs that have been reported to us by our readers. There are too many people to acknowledge individually, but to all who have written to us we are very grateful.

September 2001

William H. Press
Saul A. Teukolsky
William T. Vetterling
Brian P. Flannery

Preface to the Second Edition

Our aim in writing the original edition of *Numerical Recipes* was to provide a book that combined general discussion, analytical mathematics, algorithmics, and actual working programs. The success of the first edition puts us now in a difficult, though hardly unenviable, position. We wanted, then and now, to write a book that is informal, fearlessly editorial, unesoteric, and above all useful. There is a danger that, if we are not careful, we might produce a second edition that is weighty, balanced, scholarly, and boring.

It is a mixed blessing that we know more now than we did six years ago. Then, we were making educated guesses, based on existing literature and our own research, about which numerical techniques were the most important and robust. Now, we have the benefit of direct feedback from a large reader community. Letters to our alter-ego enterprise, Numerical Recipes Software, are in the thousands per year. (Please, *don't telephone* us.) Our post office box has become a magnet for letters pointing out that we have omitted some particular technique, well known to be important in a particular field of science or engineering. We value such letters, and digest them carefully, especially when they point us to specific references in the literature.

The inevitable result of this input is that this Second Edition of *Numerical Recipes* is substantially larger than its predecessor, in fact about 50% larger both in words and number of included programs (the latter now numbering well over 300). "Don't let the book grow in size," is the advice that we received from several wise colleagues. We have tried to follow the intended spirit of that advice, even as we violate the letter of it. We have not lengthened, or increased in difficulty, the book's principal discussions of mainstream topics. Many new topics are presented at this same accessible level. Some topics, both from the earlier edition and new to this one, are now set in smaller type that labels them as being "advanced." The reader who ignores such advanced sections completely will not, we think, find any lack of continuity in the shorter volume that results.

Here are some highlights of the new material in this Second Edition:

- a new chapter on integral equations and inverse methods
- a detailed treatment of multigrid methods for solving elliptic partial differential equations
- routines for band diagonal linear systems
- improved routines for linear algebra on sparse matrices
- Cholesky and QR decomposition
- orthogonal polynomials and Gaussian quadratures for arbitrary weight functions
- methods for calculating numerical derivatives
- Padé approximants, and rational Chebyshev approximation
- Bessel functions, and modified Bessel functions, of fractional order; and several other new special functions
- improved random number routines
- quasi-random sequences
- routines for adaptive and recursive Monte Carlo integration in high-dimensional spaces
- globally convergent methods for sets of nonlinear equations

- simulated annealing minimization for continuous control spaces
- fast Fourier transform (FFT) for real data in two and three dimensions
- fast Fourier transform (FFT) using external storage
- improved fast cosine transform routines
- wavelet transforms
- Fourier integrals with upper and lower limits
- spectral analysis on unevenly sampled data
- Savitzky-Golay smoothing filters
- fitting straight line data with errors in both coordinates
- a two-dimensional Kolmogorov-Smirnoff test
- the statistical bootstrap method
- embedded Runge-Kutta-Fehlberg methods for differential equations
- high-order methods for stiff differential equations
- a new chapter on "less-numerical" algorithms, including Huffman and arithmetic coding, arbitrary precision arithmetic, and several other topics.

Consult the Preface to the First Edition, following, or the table of Contents, for a list of the more "basic" subjects treated.

Acknowledgments

It is not possible for us to list by name here all the readers who have made useful suggestions; we are grateful for these. In the text, we attempt to give specific attribution for ideas that appear to be original, and are not known in the literature. We apologize in advance for any omissions.

Some readers and colleagues have been particularly generous in providing us with ideas, comments, suggestions, and programs for this Second Edition. We especially want to thank George Rybicki, Philip Pinto, Peter Lepage, Robert Lupton, Douglas Eardley, Ramesh Narayan, David Spergel, Alan Oppenheim, Sallie Baliunas, Scott Tremaine, Glennys Farrar, Steven Block, John Peacock, Thomas Loredo, Matthew Choptuik, Gregory Cook, L. Samuel Finn, P. Deuflhard, Harold Lewis, Peter Weinberger, David Syer, Richard Ferch, Steven Ebstein, Bradley Keister, and William Gould. We have been helped by Nancy Lee Snyder's mastery of a complicated TEX manuscript. We express appreciation to our editors Lauren Cowles and Alan Harvey at Cambridge University Press, and to our production editor Russell Hahn. We remain, of course, grateful to the individuals acknowledged in the Preface to the First Edition.

Special acknowledgment is due to programming consultant Seth Finkelstein, who wrote, rewrote, or influenced many of the routines in this book, as well as in its `Fortran`-language twin and the companion Example books. Our project has benefited enormously from Seth's talent for detecting, and following the trail of, even very slight anomalies (often compiler bugs, but occasionally our errors), and from his good programming sense. To the extent that this edition of *Numerical Recipes in C* has a more graceful and "C-like" programming style than its predecessor, most of the credit goes to Seth. (Of course, we accept the blame for the Fortranish lapses that still remain.)

We prepared this book for publication on DEC and Sun workstations running the UNIX operating system, and on a 486/33 PC compatible running MS-DOS 5.0/Windows 3.0. (See §1.0 for a list of additional computers used in

program tests.) We enthusiastically recommend the principal software used: GNU Emacs, TEX, Perl, Adobe Illustrator, and PostScript. Also used were a variety of C compilers – too numerous (and sometimes too buggy) for individual acknowledgment. It is a sobering fact that our standard test suite (exercising all the routines in this book) has uncovered compiler bugs in many of the compilers tried. When possible, we work with developers to see that such bugs get fixed; we encourage interested compiler developers to contact us about such arrangements.

WHP and SAT acknowledge the continued support of the U.S. National Science Foundation for their research on computational methods. D.A.R.P.A. support is acknowledged for §13.10 on wavelets.

June 1992

William H. Press
Saul A. Teukolsky
William T. Vetterling
Brian P. Flannery

Preface to the First Edition

We call this book *Numerical Recipes* for several reasons. In one sense, this book is indeed a "cookbook" on numerical computation. However, there is an important distinction between a cookbook and a restaurant menu. The latter presents choices among complete dishes in each of which the individual flavors are blended and disguised. The former — and this book — reveals the individual ingredients and explains how they are prepared and combined.

Another purpose of the title is to connote an eclectic mixture of presentational techniques. This book is unique, we think, in offering, for each topic considered, a certain amount of general discussion, a certain amount of analytical mathematics, a certain amount of discussion of algorithmics, and (most important) actual implementations of these ideas in the form of working computer routines. Our task has been to find the right balance among these ingredients for each topic. You will find that for some topics we have tilted quite far to the analytic side; this where we have felt there to be gaps in the "standard" mathematical training. For other topics, where the mathematical prerequisites are universally held, we have tilted towards more in-depth discussion of the nature of the computational algorithms, or towards practical questions of implementation.

We admit, therefore, to some unevenness in the "level" of this book. About half of it is suitable for an advanced undergraduate course on numerical computation for science or engineering majors. The other half ranges from the level of a graduate course to that of a professional reference. Most cookbooks have, after all, recipes at varying levels of complexity. An attractive feature of this approach, we think, is that the reader can use the book at increasing levels of sophistication as his/her experience grows. Even inexperienced readers should be able to use our most advanced routines as black boxes. Having done so, we hope that these readers will subsequently go back and learn what secrets are inside.

If there is a single dominant theme in this book, it is that practical methods of numerical computation can be simultaneously efficient, clever, and — important — clear. The alternative viewpoint, that efficient computational methods must necessarily be so arcane and complex as to be useful only in "black box" form, we firmly reject.

Our purpose in this book is thus to open up a large number of computational black boxes to your scrutiny. We want to teach you to take apart these black boxes and to put them back together again, modifying them to suit your specific needs. We assume that you are mathematically literate, i.e., that you have the normal mathematical preparation associated with an undergraduate degree in a physical science, or engineering, or economics, or a quantitative social science. We assume that you know how to program a computer. We do not assume that you have any prior formal knowledge of numerical analysis or numerical methods.

The scope of *Numerical Recipes* is supposed to be "everything up to, but not including, partial differential equations." We honor this in the breach: First, we *do* have one introductory chapter on methods for partial differential equations (Chapter 19). Second, we obviously cannot include *everything* else. All the so-called "standard" topics of a numerical analysis course have been included in this book:

linear equations (Chapter 2), interpolation and extrapolation (Chaper 3), integration (Chaper 4), nonlinear root-finding (Chapter 9), eigensystems (Chapter 11), and ordinary differential equations (Chapter 16). Most of these topics have been taken beyond their standard treatments into some advanced material which we have felt to be particularly important or useful.

Some other subjects that we cover in detail are not usually found in the standard numerical analysis texts. These include the evaluation of functions and of particular special functions of higher mathematics (Chapters 5 and 6); random numbers and Monte Carlo methods (Chapter 7); sorting (Chapter 8); optimization, including multidimensional methods (Chapter 10); Fourier transform methods, including FFT methods and other spectral methods (Chapters 12 and 13); two chapters on the statistical description and modeling of data (Chapters 14 and 15); and two-point boundary value problems, both shooting and relaxation methods (Chapter 17).

The programs in this book are included in ANSI-standard C. Versions of the book in `Fortran`, `Pascal`, and `BASIC` are available separately. We have more to say about the C language, and the computational environment assumed by our routines, in §1.1 (Introduction).

Acknowledgments

Many colleagues have been generous in giving us the benefit of their numerical and computational experience, in providing us with programs, in commenting on the manuscript, or in general encouragement. We particularly wish to thank George Rybicki, Douglas Eardley, Philip Marcus, Stuart Shapiro, Paul Horowitz, Bruce Musicus, Irwin Shapiro, Stephen Wolfram, Henry Abarbanel, Larry Smarr, Richard Muller, John Bahcall, and A.G.W. Cameron.

We also wish to acknowledge two individuals whom we have never met: Forman Acton, whose 1970 textbook *Numerical Methods that Work* (New York: Harper and Row) has surely left its stylistic mark on us; and Donald Knuth, both for his series of books on *The Art of Computer Programming* (Reading, MA: Addison-Wesley), and for TEX, the computer typesetting language which immensely aided production of this book.

Research by the authors on computational methods was supported in part by the U.S. National Science Foundation.

October 1985

William H. Press
Brian P. Flannery
Saul A. Teukolsky
William T. Vetterling

License Information

Read this section if you want to use the programs in this book on a computer. You'll need to read the following Disclaimer of Warranty, get the programs onto your computer, and acquire a Numerical Recipes software license. (Without this license, which can be the free "immediate license" under terms described below, the book is intended as a text and reference book, for reading purposes only.)

Disclaimer of Warranty

We make no warranties, express or implied, that the programs contained in this volume are free of error, or are consistent with any particular standard of merchantability, or that they will meet your requirements for any particular application. They should not be relied on for solving a problem whose incorrect solution could result in injury to a person or loss of property. If you do use the programs in such a manner, it is at your own risk. The authors and publisher disclaim all liability for direct or consequential damages resulting from your use of the programs.

How to Get the Code onto Your Computer

Pick one of the following methods:

- You can type the programs from this book directly into your computer. In this case, the *only* kind of license available to you is the free "immediate license" (see below). You are not authorized to transfer or distribute a machine-readable copy to any other person, nor to have any other person type the programs into a computer on your behalf. We do not want to hear bug reports from you if you choose this option, because experience has shown that *virtually all* reported bugs in such cases are typing errors!

- You can download the Numerical Recipes programs electronically from the Numerical Recipes On-Line Software Store, located at our Web site `http://www.nr.com`. All the files (Recipes and demonstration programs) are packaged as a single compressed file. You'll need to purchase a license to download and unpack them. Any number of single-screen licenses can be purchased instantly (with discount for multiple screens) from the On-Line Store, with fees that depend on your operating system (Windows or Macintosh versus Linux or UNIX) and whether you are affiliated with an educational institution. Purchasing a single-screen license is also the way to start if you want to acquire a more general (site or corporate) license; your single-screen cost will be subtracted from the cost of any later license upgrade.

- You can purchase media containing the programs from Cambridge University Press. A CD-ROM version in ISO-9660 format for Windows and Macintosh systems contains the complete C++ software, and also the previously available C version. The CD-ROM is available with a single-screen license for Windows or Macintosh (order ISBN 0 521 750377). More extensive CD-ROMs in ISO-9660 format for Windows, Macintosh, and UNIX/Linux systems are also available; these include the C++, C, and Fortran versions on a single CD-ROM (as well as versions in Pascal and BASIC from the first edition). These CD-ROMs are available with a single-screen license for Windows or Macintosh (order ISBN 0 521 750350), or (at a slightly higher price) with a single-screen license for UNIX/Linux workstations (order ISBN 0 521 750369). Orders for media from Cambridge University Press can be placed at 800 872-7423 (North America only) or by email to orders@cup.org (North America) or directcustserv@cambridge.org (rest of world). Or, visit the Web site `http://www.cambridge.org`.

Types of License Offered

Here are the types of licenses that we offer. Note that some types are automatically acquired with the purchase of media from Cambridge University Press, or of an unlocking password from the Numerical Recipes On-Line Software Store, while other types of licenses require that you communicate specifically with Numerical Recipes Software (email: orders@nr.com or fax: 781 863-1739). Our Web site `http://www.nr.com` has additional information.

- ["Immediate License"] If you are the individual owner of a copy of this book and you type one or more of its routines into your computer, we authorize you to use them on that computer for your own personal and noncommercial purposes. You are not authorized to transfer or distribute machine-readable copies to any other person, or to use the routines on more than one machine, or to distribute executable programs containing our routines. This is the only free license.

- ["Single-Screen License"] This is the most common type of low-cost license, with terms governed by our Single-Screen (Shrinkwrap) License document (complete terms available through our Web site). Basically, this license lets you use *Numerical Recipes* routines on any one screen (laptop, workstation, X-terminal, etc.). You may also, under this license, transfer precompiled, executable programs incorporating our routines to other, unlicensed, screens or computers, providing that (i) your application is noncommercial (i.e., does not involve the selling of your program for a fee), (ii) the programs were first developed, compiled, and successfully run on a licensed screen, and (iii) our routines are bound into the programs in such a manner that they cannot be accessed as individual routines and cannot practicably be unbound and used in other programs. That is, under this license, your program user must not be able to use our programs as part of a program library or "mix-and-match" workbench. Conditions for

other types of commercial or noncommercial distribution may be found on our Web site (`http://www.nr.com`).

- ["Multi-Screen, Server, Site, and Corporate Licenses"] The terms of the Single-Screen License can be extended to designated groups of machines, defined by number of screens, number of machines, locations, or ownership. Significant discounts from the corresponding single-screen prices are available when the estimated number of screens exceeds 40. Contact Numerical Recipes Software (email: orders@nr.com or fax: 781 863-1739) for details.

- ["Course Right-to-Copy License"] Instructors at accredited educational institutions who have adopted this book for a course, and who have already purchased a Single-Screen License (either acquired with the purchase of media, or from the Numerical Recipes On-Line Software Store), may license the programs for use in that course as follows: Mail your name, title, and address; the course name, number, dates, and estimated enrollment; and advance payment of $5 per (estimated) student to Numerical Recipes Software, at this address: P.O. Box 380243, Cambridge, MA 02238-0243 (USA). You will receive by return mail a license authorizing you to make copies of the programs for use by your students, and/or to transfer the programs to a machine accessible to your students (but only for the duration of the course).

About Copyrights on Computer Programs

Like artistic or literary compositions, computer programs are protected by copyright. Generally it is an infringement for you to copy into your computer a program from a copyrighted source. (It is also not a friendly thing to do, since it deprives the program's author of compensation for his or her creative effort.) Under copyright law, all "derivative works" (modified versions, or translations into another computer language) also come under the same copyright as the original work.

Copyright does not protect ideas, but only the expression of those ideas in a particular form. In the case of a computer program, the ideas consist of the program's methodology and algorithm, including the necessary sequence of steps adopted by the programmer. The expression of those ideas is the program source code (particularly any arbitrary or stylistic choices embodied in it), its derived object code, and any other derivative works.

If you analyze the ideas contained in a program, and then express those ideas in your own completely different implementation, then that new program implementation belongs to you. That is what we have done for those programs in this book that are not entirely of our own devising. When programs in this book are said to be "based" on programs published in copyright sources, we mean that the ideas are the same. The expression of these ideas as source code is our own. We believe that no material in this book infringes on an existing copyright.

Trademarks

Several registered trademarks appear within the text of this book: Sun, Solaris, Ultra, and WorkShop are trademarks of Sun Microsystems, Inc. Microsoft, Windows, and Visual C++ are trademarks of Microsoft Corporation. DEC and VMS are trademarks of Compaq Computer Corporation. IBM, AIX, and RS/6000 are trademarks of International Business Machines Corporation. Intel, Pentium and KAI C++ are trademarks of Intel Corporation. Linux is a trademark of Linus Torvalds. Apple and Macintosh are trademarks of Apple Computer, Inc. Borland C++ Builder is a trademark of Inprise Corporation. UNIX is a trademark of The Open Group. IMSL is a trademark of Visual Numerics, Inc. NAG refers to proprietary computer software of Numerical Algorithms Group (USA) Inc. PostScript and Adobe Illustrator are trademarks of Adobe Systems Incorporated. Last, and no doubt least, Numerical Recipes, NR, and nr.com (when identifying our products) are trademarks of Numerical Recipes Software.

Attributions

The fact that ideas are legally "free as air" in no way supersedes the ethical requirement that ideas be credited to their known originators. When programs in this book are based on known sources, whether copyrighted or in the public domain, published or "handed-down," we have attempted to give proper attribution. Unfortunately, the lineage of many programs in common circulation is often unclear. We would be grateful to readers for new or corrected information regarding attributions, which we will attempt to incorporate in subsequent printings.

Computer Programs by Chapter and Section

1.0	`flmoon`	calculate phases of the moon by date
1.1	`julday`	Julian Day number from calendar date
1.1	`badluk`	Friday the 13th when the moon is full
1.1	`caldat`	calendar date from Julian day number
2.1	`gaussj`	Gauss-Jordan matrix inversion and linear equation solution
2.3	`ludcmp`	linear equation solution, LU decomposition
2.3	`lubksb`	linear equation solution, backsubstitution
2.4	`tridag`	solution of tridiagonal systems
2.4	`banmul`	multiply vector by band diagonal matrix
2.4	`bandec`	band diagonal systems, decomposition
2.4	`banbks`	band diagonal systems, backsubstitution
2.5	`mprove`	linear equation solution, iterative improvement
2.6	`svbksb`	singular value backsubstitution
2.6	`svdcmp`	singular value decomposition of a matrix
2.6	`pythag`	calculate $(a^2 + b^2)^{1/2}$ without overflow
2.7	`cyclic`	solution of cyclic tridiagonal systems
2.7	`sprsin`	convert matrix to sparse format
2.7	`sprsax`	product of sparse matrix and vector
2.7	`sprstx`	product of transpose sparse matrix and vector
2.7	`sprstp`	transpose of sparse matrix
2.7	`sprspm`	pattern multiply two sparse matrices
2.7	`sprstm`	threshold multiply two sparse matrices
2.7	`linbcg`	biconjugate gradient solution of sparse systems
2.7	`snrm`	used by `linbcg` for vector norm
2.7	`atimes`	used by `linbcg` for sparse multiplication
2.7	`asolve`	used by `linbcg` for preconditioner
2.8	`vander`	solve Vandermonde systems
2.8	`toeplz`	solve Toeplitz systems
2.9	`choldc`	Cholesky decomposition
2.9	`cholsl`	Cholesky backsubstitution
2.10	`qrdcmp`	QR decomposition
2.10	`qrsolv`	QR backsubstitution
2.10	`rsolv`	right triangular backsubstitution
2.10	`qrupdt`	update a QR decomposition
2.10	`rotate`	Jacobi rotation used by `qrupdt`
3.1	`polint`	polynomial interpolation
3.2	`ratint`	rational function interpolation
3.3	`spline`	construct a cubic spline
3.3	`splint`	cubic spline interpolation
3.4	`locate`	search an ordered table by bisection

3.4	`hunt`	search a table when calls are correlated
3.5	`polcoe`	polynomial coefficients from table of values
3.5	`polcof`	polynomial coefficients from table of values
3.6	`polin2`	two-dimensional polynomial interpolation
3.6	`bcucof`	construct two-dimensional bicubic
3.6	`bcuint`	two-dimensional bicubic interpolation
3.6	`splie2`	construct two-dimensional spline
3.6	`splin2`	two-dimensional spline interpolation
4.2	`trapzd`	trapezoidal rule
4.2	`qtrap`	integrate using trapezoidal rule
4.2	`qsimp`	integrate using Simpson's rule
4.3	`qromb`	integrate using Romberg adaptive method
4.4	`midpnt`	extended midpoint rule
4.4	`qromo`	integrate using open Romberg adaptive method
4.4	`midinf`	integrate a function on a semi-infinite interval
4.4	`midsql`	integrate a function with lower square-root singularity
4.4	`midsqu`	integrate a function with upper square-root singularity
4.4	`midexp`	integrate a function that decreases exponentially
4.5	`qgaus`	integrate a function by Gaussian quadratures
4.5	`gauleg`	Gauss-Legendre weights and abscissas
4.5	`gaulag`	Gauss-Laguerre weights and abscissas
4.5	`gauher`	Gauss-Hermite weights and abscissas
4.5	`gaujac`	Gauss-Jacobi weights and abscissas
4.5	`gaucof`	quadrature weights from orthogonal polynomials
4.5	`orthog`	construct nonclassical orthogonal polynomials
4.6	`quad3d`	integrate a function over a three-dimensional space
5.1	`eulsum`	sum a series by Euler–van Wijngaarden algorithm
5.3	`ddpoly`	evaluate a polynomial and its derivatives
5.3	`poldiv`	divide one polynomial by another
5.3	`ratval`	evaluate a rational function
5.7	`dfridr`	numerical derivative by Ridders' method
5.8	`chebft`	fit a Chebyshev polynomial to a function
5.8	`chebev`	Chebyshev polynomial evaluation
5.9	`chder`	derivative of a function already Chebyshev fitted
5.9	`chint`	integrate a function already Chebyshev fitted
5.10	`chebpc`	polynomial coefficients from a Chebyshev fit
5.10	`pcshft`	polynomial coefficients of a shifted polynomial
5.11	`pccheb`	inverse of `chebpc`; use to economize power series
5.12	`pade`	Padé approximant from power series coefficients
5.13	`ratlsq`	rational fit by least-squares method
6.1	`gammln`	logarithm of gamma function
6.1	`factrl`	factorial function
6.1	`bico`	binomial coefficients function
6.1	`factln`	logarithm of factorial function

6.1	beta	beta function
6.2	gammp	incomplete gamma function
6.2	gammq	complement of incomplete gamma function
6.2	gser	series used by gammp and gammq
6.2	gcf	continued fraction used by gammp and gammq
6.2	erff	error function
6.2	erffc	complementary error function
6.2	erfcc	complementary error function, concise routine
6.3	expint	exponential integral E_n
6.3	ei	exponential integral Ei
6.4	betai	incomplete beta function
6.4	betacf	continued fraction used by betai
6.5	bessj0	Bessel function J_0
6.5	bessy0	Bessel function Y_0
6.5	bessj1	Bessel function J_1
6.5	bessy1	Bessel function Y_1
6.5	bessy	Bessel function Y of general integer order
6.5	bessj	Bessel function J of general integer order
6.6	bessi0	modified Bessel function I_0
6.6	bessk0	modified Bessel function K_0
6.6	bessi1	modified Bessel function I_1
6.6	bessk1	modified Bessel function K_1
6.6	bessk	modified Bessel function K of integer order
6.6	bessi	modified Bessel function I of integer order
6.7	bessjy	Bessel functions of fractional order
6.7	beschb	Chebyshev expansion used by bessjy
6.7	bessik	modified Bessel functions of fractional order
6.7	airy	Airy functions
6.7	sphbes	spherical Bessel functions j_n and y_n
6.8	plgndr	Legendre polynomials, associated (spherical harmonics)
6.9	frenel	Fresnel integrals $S(x)$ and $C(x)$
6.9	cisi	cosine and sine integrals Ci and Si
6.10	dawson	Dawson's integral
6.11	rf	Carlson's elliptic integral of the first kind
6.11	rd	Carlson's elliptic integral of the second kind
6.11	rj	Carlson's elliptic integral of the third kind
6.11	rc	Carlson's degenerate elliptic integral
6.11	ellf	Legendre elliptic integral of the first kind
6.11	elle	Legendre elliptic integral of the second kind
6.11	ellpi	Legendre elliptic integral of the third kind
6.11	sncndn	Jacobian elliptic functions
6.12	hypgeo	complex hypergeometric function
6.12	hypser	complex hypergeometric function, series evaluation
6.12	hypdrv	complex hypergeometric function, derivative of
7.1	ran0	random deviate by Park and Miller minimal standard
7.1	ran1	random deviate, minimal standard plus shuffle

7.1	ran2	random deviate by L'Ecuyer long period plus shuffle
7.1	ran3	random deviate by Knuth subtractive method
7.2	expdev	exponential random deviates
7.2	gasdev	normally distributed random deviates
7.3	gamdev	gamma-law distribution random deviates
7.3	poidev	Poisson distributed random deviates
7.3	bnldev	binomial distributed random deviates
7.4	irbit1	random bit sequence
7.4	irbit2	random bit sequence
7.5	psdes	"pseudo-DES" hashing of 64 bits
7.5	ran4	random deviates from DES-like hashing
7.7	sobseq	Sobol's quasi-random sequence
7.8	vegas	adaptive multidimensional Monte Carlo integration
7.8	rebin	sample rebinning used by vegas
7.8	miser	recursive multidimensional Monte Carlo integration
7.8	ranpt	get random point, used by miser
8.1	piksrt	sort an array by straight insertion
8.1	piksr2	sort two arrays by straight insertion
8.1	shell	sort an array by Shell's method
8.2	sort	sort an array by quicksort method
8.2	sort2	sort two arrays by quicksort method
8.3	hpsort	sort an array by heapsort method
8.4	indexx	construct an index for an array
8.4	sort3	sort, use an index to sort 3 or more arrays
8.4	rank	construct a rank table for an array
8.5	select	find the Nth largest in an array
8.5	selip	find the Nth largest, without altering an array
8.5	hpsel	find M largest values, without altering an array
8.6	eclass	determine equivalence classes from list
8.6	eclazz	determine equivalence classes from procedure
9.0	scrsho	graph a function to search for roots
9.1	zbrac	outward search for brackets on roots
9.1	zbrak	inward search for brackets on roots
9.1	rtbis	find root of a function by bisection
9.2	rtflsp	find root of a function by false-position
9.2	rtsec	find root of a function by secant method
9.2	zriddr	find root of a function by Ridders' method
9.3	zbrent	find root of a function by Brent's method
9.4	rtnewt	find root of a function by Newton-Raphson
9.4	rtsafe	find root of a function by Newton-Raphson and bisection
9.5	laguer	find a root of a polynomial by Laguerre's method
9.5	zroots	roots of a polynomial by Laguerre's method with deflation
9.5	zrhqr	roots of a polynomial by eigenvalue methods
9.5	qroot	complex or double root of a polynomial, Bairstow