天元基金 影印系列丛书

Mats Andersson 著

复分析中的若干论题

Topics in Complex Analysis

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Preface

This book is an outgrowth of lectures given on several occasions at Chalmers University of Technology and Göteborg University during the last ten years. As opposed to most introductory books on complex analysis, this one assumes that the reader has previous knowledge of basic real analysis. This makes it possible to follow a rather quick route through the most fundamental material on the subject in order to move ahead to reach some classical highlights (such as Fatou theorems and some Nevanlinna theory), as well as some more recent topics (for example, the corona theorem and the H^1-BMO duality) within the time frame of a one-semester course. Sections 3 and 4 in Chapter 2, Sections 5 and 6 in Chapter 3, Section 3 in Chapter 5, and Section 4 in Chapter 7 were not contained in my original lecture notes and therefore might be considered special topics. In addition, they are completely independent and can be omitted with no loss of continuity.

The order of the topics in the exposition coincides to a large degree with historical developments. The first five chapters essentially deal with theory developed in the nineteenth century, whereas the remaining chapters contain material from the early twentieth century up to the 1980s.

Choosing methods of presentation and proofs is a delicate task. My aim has been to point out connections with real analysis and harmonic analysis, while at the same time treating classical complex function theory. I also have tried to present some general tools that can be of use in other areas of analysis. Whereas these various aims sometimes can be incompatible, at times the scope of the book imposes some natural restrictions. For example, Runge's theorem is proved by the "Hahn–Banach method," partly because it is probably the simplest way to do so, but also because it is a technique that every student in analysis should become familiar with. However, a constructive proof is outlined as an exercise. Complex analysis is one of the origins of harmonic analysis, and several results in the latter subject have forerunners in complex analysis. Fatou's theorem in Chapter 6 is proved using standard harmonic analysis, in particular using the weak-type estimate for the Hardy–Littlewood maximal function. However, most standard tools from harmonic analysis are beyond the scope of this book, and therefore, the L^p-boundedness of the Hilbert transform and the H^p-space theory, for instance, are treated with complex analytic methods. Carleson's inequality is proved by an elementary argument due to B. Berndtsson, rather than using the L^p-estimate for the maximal function, and Carleson's interpolation theorem is proved using the beautiful and explicit construction of the interpolating function due to P. Jones from the 1980s. However, a proof based on the $\bar{\partial}_b$-equation is indicated in an exercise.

Necessary prerequisites for the reader are basic courses in integration theory and functional analysis. In the text, I sometimes refer to distribution theory, but this is merely for illustration and can be skipped over with no serious loss of understanding. The reader whose memory of an elementary (undergraduate) course in complex analysis is not so strong is advised to consult an appropriate text for supplementary reading.

As usual, the exercises can be divided into two categories: those that merely test the reader's understanding of or shed light on definitions and theorems (these are sometimes interposed in the text) and those that ask the reader to apply the theory or to develop it further. I think that for optimal results a good deal of the time reserved for the study of this subject should be devoted to grappling with the exercises. The exercises follow the approximate order of topics in the corresponding chapters, and thus, the degree of difficulty can vary greatly. For some of the exercises, I have supplied hints and answers.

At the end of each chapter, I have included references to the main results, usually to some more encyclopedic treatment of the subject in question, but sometimes to original papers. If references do not always appear, this is solely for the sake of expediency and does not imply any claim of originality on my part. My contribution consists mainly in the disposition and adaptation of some material and proofs, previously found only in papers or encyclopedic texts addressed to experts, into a form that hopefully will be accessible to students.

Finally, I would like to take this opportunity to express my appreciation to all of the students and colleagues who have pointed out errors and obscurities in various earlier versions of the manuscript and made valuable suggestions for improvements. For their help with the final version, I would like to thank in particular Lars Alexandersson, Bo Berndtsson, Hasse Carlsson, Niklas Lindholm, and Jeffrey Steif.

Göteborg, Sweden Mats Andersson

Contents

Preliminaries

§1. Notation

Throughout this book the letters Ω and K always will denote open and compact sets, respectively, in $\mathbb{R}^2$, and ω will denote a bounded open set with (when necessary) piecewise C^1 boundary $\partial\omega$, which is always supposed to be positively oriented; i.e., one has ω on the left-hand side when passing along $\partial\omega$. The notation $\omega \subset\subset \Omega$ means that the closure of ω is a compact subset of Ω, and $d(K, E)$ denotes the distance between the sets K and E. Moreover, $D(a, r)$ is the open disk with center at a and radius r, and U denotes the unit disk, i.e., $U = D(0, 1)$, and T is its boundary $\partial U = \{z;\ |z| = 1\}$. The closure of a set $E \in \mathbb{R}^2$ is denoted by $\overline{E}$ and its interior is denoted by $\operatorname{int} E$.

The space of k times (real) differentiable (complex valued) functions in Ω is denoted by $C^k(\Omega)$ (however, we write $C(\Omega)$ rather than $C^0(\Omega)$) and $C^\infty(\Omega) = \cap C^k(\Omega)$. Moreover, $C^k(\overline{\Omega})$ is the subspace of functions in $C^k(\Omega)$ whose derivatives up to the kth order have continuous extensions to $\overline{\Omega}$, and $C_0^k(\Omega)$ is the subspace of functions in $C^k(\Omega)$ that have compact support in Ω. Lebesgue measure in $\mathbb{R}^2$ is denoted by $d\lambda$, whereas $d\sigma$ denotes arc length along curves. We use the standard abbreviation a.e. for "almost every(where)." We also use u.c. for "uniformly on compact sets." If f, ϕ are functions, then "$f = \mathcal{O}(\phi)$ when $x \to a$" means that f/ϕ is bounded in a neighborhood of a and "$f = o(\phi)$ when $x \to a$" means that $f/\phi \to 0$ when $x \to a$. Sometimes we also use the notation $f \lesssim g$, which means that f is less than or equal to some constant times g. Moreover, $f \sim g$ stands for $f \lesssim g$ and $f \gtrsim g$.

We will use standard facts from basic courses in integration theory and functional analysis. Sometimes we also refer to distribution theory (mainly in remarks), but these comments are meant merely for illustration and always can be passed over with no loss of continuity. In the next section we have assembled some facts that will be used frequently in the text. In the first chapters we refer to them explicitly but later on often only implicitly.

Almost all necessary background material can be found in [F] or [Ru1], combined with a basic calculus book. For the facts in item B below see, e.g., [Hö], which also serves as a general reference on distribution theory.

§2. Some Facts

A. Some facts from calculus. If f is a map from Ω into $\mathbb{R}^2$ that is C^1 in a neighborhood of a, then

$$f(a+x) = f(a) + Df|_a x + o(|x|) \quad \text{when} \quad x \to 0,$$

for some linear map $x \mapsto Df|_a x$, i.e., f is *differentiable* at a. If f is considered as a complex valued function, then

$$Df|_a x = x_1 \frac{\partial f}{\partial x_1}\Big|_a + x_2 \frac{\partial f}{\partial x_2}\Big|_a.$$

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $a \le t \le b$ be a piecewise C^1 parametrization of the curve Γ. If P, Q are continuous functions on Γ, then

$$\int_\Gamma P dx + Q dy = \int_a^b (P(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + Q(\gamma_1(t), \gamma_2(t))\gamma_2'(t))\, dt,$$

and this expression is independent of the choice of parametrization. Note that

$$\int_\Gamma f dg = \int_a^b f \circ \gamma \frac{d(g \circ \gamma)}{dt} dt$$

if $f, g \in C^1(\Omega)$ and $\Gamma \subset \Omega$. In particular, for an exact form we have

$$\int_\Gamma dg = g(\gamma(b)) - g(\gamma(a)).$$

The *arc length* of the curve Γ is

$$|\Gamma| = \int_\Gamma d\sigma = \int_a^b |\gamma'(t)|^2 dt = \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} dt$$

and

$$\left|\int_\Gamma P dx + Q dy\right| \le \int_\Gamma \sqrt{|P|^2 + |Q|^2} d\sigma \le |\Gamma| \sup_\Gamma \sqrt{|P|^2 + |Q|^2}.$$

Green's formula (Stokes' theorem) states that if $P, Q \in C^1(\overline{\omega})$, then

$$\int_{\partial\omega} P dx + Q dy = \int_\omega (Q_x - P_y) d\lambda,$$

whereas Green's identity (Green's formula) states that if $u, v \in C^1(\overline{\omega})$, then

$$\int_\omega (u\Delta v - v\Delta u) d\lambda = \int_{\partial\omega} \left(u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}\right) d\sigma,$$

where $\partial/\partial\eta$ is the outward normal derivative, i.e., $\partial u/\partial\eta = \sum \eta_j(\partial u/\partial x_j)$ if $\eta = (\eta_1, \eta_2)$ is the outward normal to $\partial\omega$.

On some occasions we also refer to the inverse function theorem, see, e.g., [Hö]: If $f: \Omega \to \mathbb{R}^2$ is C^1 and its derivauive $Df|_a$ at $a \in \Omega$ is nonsingular, then locally f has a C^1 inverse g.

B. Existence of test functions. There are "plenty" of functions in $C_0^\infty(\Omega)$, namely,

(i) for any $K \subset \Omega$ there is a $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ in a neighborhood of K and $0 \le \phi \le 1$.

(ii) if $f \in L^1_{\text{loc}}(\Omega)$ and $\int_\Omega \phi f d\lambda = 0$ for all $\phi \in C_0^\infty(\Omega)$, then $f = 0$ a.e.

(iii) if f is continuous on $K \subset \Omega$ and $\epsilon > 0$, there is a $\phi \in C_0^\infty(\Omega)$ such that $\sup_K |\phi - f| < \epsilon$.

(iv) if $\cup \Omega_\alpha = \Omega$, then there is a *smooth partition of unity subordinate to the open cover* Ω_α, i.e., there are $\phi_k \in C_0^\infty(\Omega_{\alpha_k})$ such that $0 \le \phi_k \le 1$, locally only a finite number of ϕ_k are nonvanishing and $\sum_k \phi_k \equiv 1$ in Ω.

C. Integration theory. From integration theory we use the standard convergence theorems, such as Fatou's lemma, Lebesgue's theorem on dominated convergence, and the monotone convergence theorem. Moreover, we frequently use Jensen's and Hölder's inequalities and Fubini's theorem, the duality of L^p and L^q for $p < \infty$, and the one-to-one correspondence between the continuous linear functionals on $C(K)$ and the space of finite complex Borel measures on K (usually just referred to as measures on K). Furthermore, we need the Jordan decomposition of a real measure, the Lebesgue–Radon–Nikodym decomposition of a complex measure with respect to a positive measure (the Lebesgue measure in our case) and the weak type 1-1 estimate for the Hardy–Littlewood maximal function.

In particular, we frequently will make use of "differentiation under the integral sign": Suppose that X, μ is a measure space and $f(x,t)$ is a measurable function on $X \times I$, where I is an interval in $\mathbb{R}$, which is continuously differentiable in t. Suppose further that $f(x,t)$ and $f'(x,t)$ are in $L^1(\mu)$ for each fixed t so that

$$g(t) = \int f(x,t) d\mu(x) \quad \text{and} \quad h(t) = \int f'(x,t) d\mu(x)$$

are well-defined. One may ask whether $g'(t) = h(t)$. Suppose that there is a $\psi \in L^1(\mu)$ such that

$$|f'(x,t)| \le \psi(x).$$

Then $h(t)$ is continuous by the dominated convergence theorem. Moreover,

$$\iint_{X \times I} |f'(x,t)| d\mu(x) dt < \infty,$$

so we can use Fubini's theorem:

$$\int_a^b h(t) dt = \int_a^b \left(\int f'(x,t) d\mu(x) \right) dt = \int \left(\int_a^b f'(x,t) dt \right) d\mu(x)$$

and hence

$$\int_a^b h(t)dt = \int (f(x,b) - f(x,a))\, d\mu(x) = g(b) - g(a)$$

for all $a, b \in I$. Since h is continuous, $g' = h$.

D. Functional analysis. We will use basic results such as orthogonal decomposition and Parseval's equality in Hilbert spaces, the Hahn–Banach theorem, the Banach–Steinhaus theorem, and the open mapping theorem in Banach spaces. Moreover, on some occasions we require Arzela–Ascoli's theorem on locally equicontinuous subsets of $C(\Omega)$ and Tietze's extension theorem.

We also refer to the Fourier transform, Plancherel's formula, and the inversion formula; see, e.g., Ch. 9 in [Ru1].

1

Some Basic Properties of Analytic Functions

§1. Definition and Integral Representation

We identify $\mathbb{C}$ with $\mathbb{R}^2$ by identifying the complex number $z = x + iy$ with the point $(x, y) \in \mathbb{R}^2$. Observe that a (complex-valued) differential form $P dx + Q dy$ always can be written in the form $f dz + g d\bar{z}$, where $dz = dx + i dy$ and $d\bar{z} = dx - i dy$ (take $f = (P - iQ)/2$ and $g = (P + iQ)/2$). This motivates us to introduce the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

so that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}. \tag{1.1}$$

Note that $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 = 4\partial^2/\partial z \partial \bar{z}$.

1.1 Proposition. *If f is differentiable at the point a, then the limit*

$$\lim_{z \to 0} \frac{f(a + z) - f(a)}{z} \tag{1.2}$$

exists if and only if $(\partial f/\partial \bar{z})|_a = 0$; and in that case, the limit equals $(\partial f/\partial z)|_a$.

The limit (if it exists) is denoted by $f'(a)$.

Proof. The differentiability of f is equivalent to

$$f(a + z) - f(a) = z\frac{\partial f}{\partial z}\bigg|_a + \bar{z}\frac{\partial f}{\partial \bar{z}}\bigg|_a + o(|z|).$$

Thus, (1.2) exists if and only if

$$\lim_{z\to 0} \frac{\bar z}{z} \frac{\partial f}{\partial \bar z}\Big|_a$$

exists, and this holds if and only if $(\partial f/\partial \bar z)|_a = 0$. The last statement follows immediately. □

Notice that if $f = u + iv$, where u and v are real, then the *Cauchy–Riemann equation* $\partial f/\partial \bar z = 0$ is equivalent to

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \tag{1.3}$$

(identify the real and imaginary parts in the equation $(\partial/\partial x + i\partial/\partial y)(u + iv) = 0$).

Definition. A function $f \in C^1(\Omega)$ is *analytic* (or *holomorphic*) in Ω if $\partial f/\partial \bar z = 0$ in Ω. The set of analytic functions is denoted by $A(\Omega)$.

In view of Proposition 1.1, $f \in C^1(\Omega)$ is analytic in Ω if and only if (1.2) exists for all $a \in \Omega$, but we even have

1.2 Goursat's Theorem. *If f is any (complex valued) function in Ω such that (1.2) exists for all $a \in \Omega$, then f is C^1 and hence analytic.*

The proof appears later on! In most cases, it is advantageous to write the Cauchy–Riemann equation in the complex form $\partial f/\partial \bar z = 0$, rather than as (1.3). For instance, clearly the product rules

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z} g + f\frac{\partial g}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \bar z}(fg) = \frac{\partial f}{\partial \bar z} g + f\frac{\partial g}{\partial \bar z}$$

hold; thus, if $f, g \in A(\Omega)$, one immediately finds that $fg \in A(\Omega)$ and $(fg)' = f'g + fg'$. Suppose that $h(t)$ is C^1 on an interval $I \subset \mathbb{R}$ and that f is C^1 in a neighborhood of the image of h in $\mathbb{C}$. Then by (1.1),

$$\frac{d(f\circ h)}{dt}dt = d(f\circ h) = \frac{\partial f}{\partial z}dh + \frac{\partial f}{\partial \bar z}d\bar h = \frac{\partial f}{\partial z}\frac{dh}{dt}dt + \frac{\partial f}{\partial \bar z}\frac{d\bar h}{dt}dt,$$

and therefore we have the chain rule

$$\frac{df\circ h}{dt} = \frac{\partial f}{\partial z}\frac{dh}{dt} + \frac{\partial f}{\partial \bar z}\frac{d\bar h}{dt}.$$

In the same way, if $h(\zeta)$ is C^1 in some domain in $\mathbb{C}$, then

$$\frac{\partial f\circ h}{\partial \tau} = \frac{\partial f}{\partial z}\frac{\partial h}{\partial \tau} + \frac{\partial f}{\partial \bar z}\frac{\partial \bar h}{\partial \tau} \quad \text{and} \quad \frac{\partial f\circ h}{\partial \bar\tau} = \frac{\partial f}{\partial z}\frac{\partial h}{\partial \bar\tau} + \frac{\partial f}{\partial \bar z}\frac{\partial \bar h}{\partial \bar\tau}.$$

Thus, if f, g are analytic, then $f\circ g$ is analytic and $(f\circ g)' = f'(g)g'$.

1.3 Example. If $f \in A(\Omega)$ and $f'(z) = 0$ for all $z \in \Omega$, then $df = 0$ and hence f is locally constant. Suppose now that $f \in A(\{\rho < |z| < R\})$ and that $f(z) = f(re^{i\theta})$ only depends on θ. Then by the chain rule

$$0 = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial r} = e^{i\theta}\frac{\partial f}{\partial z} = e^{i\theta} f',$$

and therefore f is constant. The same conclusion holds if f is independent of θ.

Here are some other simple consequences of the product rule and the chain rule (and the definition).

(a) If $f, g \in A(\Omega)$ and $\alpha, \beta \in \mathbb{C}$, then $fg \in A(\Omega)$ and $\alpha f + \beta g \in A(\Omega)$.
(b) $z \mapsto z^m$, m being a natural number, is analytic in $\mathbb{C}$.
(c) If $f \in A(\Omega)$, then $1/f \in A(\Omega \setminus \{f = 0\})$. (First show that $1/z \in A(\mathbb{C} \setminus \{0\})$!)
(d) $z \mapsto e^z =_{\text{def}} e^x(\cos y + i \sin y)$ is analytic in $\mathbb{C}$.
(e) $(\partial/\partial z)z^m = mz^{m-1}$, $(\partial/\partial z)e^z = e^z$, $(\partial/\partial z)(1/f) = -f'/f^2$.

Exercise 1. Show that

(a) $\overline{\partial f/\partial \bar{z}} = \partial \bar{f}/\partial z$.
(b) if $f \in A(\Omega)$, then $z \mapsto \overline{f(\bar{z})}$ is analytic in $\{z;\ \bar{z} \in \Omega\}$.
(c) if $f \in A(\Omega)$ and f is real, then f is (locally) constant.
(d) if $f \in A(\Omega)$ and $|f|$ is constant, then f is (locally) constant.

If the curve Γ is given by $r(t) = r_1(t) + ir_2(t)$, $a \le t \le b$, then, see A in the preliminaries,

$$\int_\Gamma f dz + g d\bar{z} = \int_a^b (f(r(t))r'(t) + g(r(t))\overline{r'(t)})dt,$$

where of course $r'(t) = r_1'(t) + ir_2'(t)$. Moreover,

$$\left|\int_\Gamma f dz\right| = \left|\int_a^b f(r(t))r'(t)dt\right| \le \int_a^b |f(r(t))||r'(t)|dt = \int_\Gamma |f| d\sigma$$

so that ($|dz| = d\sigma$)

$$\left|\int_\Gamma f dz\right| \le \int_\Gamma |f||dz| \le |\Gamma| \sup_\Gamma |f|. \tag{1.4}$$

1.4 Proposition. *If $F \in A(\Omega)$ and $f = F'$, then*

$$\int_\Gamma f dz = F(r(b)) - F(r(a)).$$

In particular, $\int_\Gamma f dz = 0$ if Γ is closed.

Proof. $f dz = (\partial F/\partial z)dz = (\partial F/\partial z)dz + (\partial F/\partial \bar{z})d\bar{z} = dF$; therefore, $f dz$ is an exact form and thus the proposition follows, cf. item A in the preliminaries. □

In complex notation Green's formula becomes (check!)

$$\int_{\partial\omega} f dz + g d\bar{z} = 2i \int_{\omega} \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right) d\lambda(z), \qquad f, g \in C^1(\overline{\omega}). \tag{1.5}$$

From this we immediately get

1.5 Cauchy's Integral Theorem. *If $f \in A(\omega) \cap C^1(\overline{\omega})$, then*

$$\int_{\partial\omega} f dz = 0.$$

The next theorem in particular tells us that the values of an analytic function in the interior of a domain are determined by its values on the boundary.

1.6 Cauchy's Formula. *If $f \in C^1(\overline{\omega})$ and $z \in \omega$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\zeta) d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\omega} \frac{\partial f}{\partial \bar{\zeta}}(\zeta) \frac{d\lambda(\zeta)}{\zeta - z}.$$

In particular, if $f \in A(\omega) \cap C^1(\overline{\omega})$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

Proof. Take $z \in \omega$. For ϵ so small that $\{\zeta;\ |\zeta - z| < \epsilon\} \subset \omega$, (1.5) (with f replaced by $f(\zeta)/(\zeta - z)$) gives that

$$2i \int_{\omega \setminus \{|\zeta - z| < \epsilon\}} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\lambda(\zeta)}{\zeta - z} = \int_{\partial\omega} \frac{f d\zeta}{\zeta - z} - \int_{|\zeta - z| = \epsilon} \frac{f d\zeta}{\zeta - z}, \tag{1.6}$$

since $\zeta \to 1/(\zeta - z)$ is analytic in $\omega \setminus \{|\zeta - z| < \epsilon|\}$. By (1.5) again, we get

$$\begin{aligned}
\int_{|\zeta - z| = \epsilon} \frac{f d\zeta}{\zeta - z} &= \frac{1}{\epsilon^2} \int_{|\zeta - z| = \epsilon} (\bar{\zeta} - \bar{z}) f d\zeta \\
&= \frac{2i}{\epsilon^2} \int_{|\zeta - z| < \epsilon} \left(f(\zeta) + (\bar{\zeta} - \bar{z}) \frac{\partial f}{\partial \bar{\zeta}}\right) d\lambda(\zeta) \\
&= \frac{2\pi i}{\pi \epsilon^2} \int_{|\zeta - z| < \epsilon} \left(f(z) + \mathcal{O}(|\zeta - z|)\right) d\lambda(\zeta) = 2\pi i f(z) + \mathcal{O}(\epsilon).
\end{aligned}$$

Since $\zeta \to (\zeta - z)^{-1}$ is locally integrable, the theorem follows from (1.6) when ϵ tends to zero. □

1.7 Some Simple but Important Consequences.
(a) By Proposition 1.4 and Cauchy's formula (with $f \equiv 1$), we get

$$\int_{|\zeta| = \epsilon} \zeta^{-n} d\zeta = \begin{cases} 2\pi i & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \neq 1. \end{cases}$$

(b) If the curve Γ starts at a and ends up at b, then, by Proposition 1.4,

$$\int_\Gamma d\zeta = [\zeta]_a^b = b - a.$$

(c) If $\phi \in C_0^1(\mathbf{C})$, then (by Cauchy's formula)

$$\phi(z) = -\frac{1}{\pi}\int \frac{(\partial\phi/\partial\bar\zeta)(\zeta)d\lambda(\zeta)}{\zeta - z} = \frac{\partial}{\partial\bar z}\left(-\frac{1}{\pi}\int\frac{\phi(\zeta)d\lambda(\zeta)}{\zeta - z}\right), \tag{1.7}$$

where the second equality is obtained by making the linear change of variables $\zeta \mapsto \zeta + z$ in the last integral and differentiating under the integral sign. Hence $(\partial/\partial\bar\zeta)1/\pi\zeta = \delta_0$ (the Dirac measure) in the distribution sense; this is equivalent to $\Delta \log|\zeta|^2 = 4\pi\delta_0$ since $(\partial/\partial\zeta)\log|\zeta|^2 = 1/\zeta$ (even in the distribution sense).

(d) From Cauchy's formula it follows that analytic functions have the *mean value property*:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{it})dt.$$

To see this, one simply makes the substitution $\zeta = z + re^{it}, 0 \le t < 2\pi$ (so that $d\zeta = ire^{it}dt = i(\zeta - z)dt$) in the formula

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z| = r}\frac{f(\zeta)d\zeta}{\zeta - z}.$$

(e) If f is analytic and we differentate under the integral sign in Cauchy's formula, we find that f is C^∞, $f^{(m)}$ is analytic, and

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial\omega}\frac{f(\zeta)d\zeta}{(\zeta - z)^{m+1}}, \quad m = 0, 1, 2, \ldots. \tag{1.8}$$

Thus in particular we have that $A(\Omega) \subset C^\infty(\Omega)$ for any Ω.

1.8 Proposition. *If $K \subset \omega \subset\subset \Omega$, then there are constants $C_m = C_{m,\omega,K}$ such that for all $f \in A(\Omega)$,*

$$\sup_K |f^{(m)}| \le C_m \|f\|_{L^1(\omega)}.$$

Proof. Take $\phi \in C_0^\infty(\omega)$, $0 \le \phi \le 1$, such that $\phi \equiv 1$ in a neighborhood of K. Let $\delta = d(K, \{z \in \omega;\ \phi(z) \ne 1\})$. Since $f\phi \in C_0^\infty(\mathbf{C})$, we get by (1.7)

$$f(z) = (f\phi)(z) = -\frac{1}{\pi}\int\frac{\partial\phi}{\partial\bar\zeta}\frac{f(\zeta)d\lambda(\zeta)}{\zeta - z}, \quad z \in K. \tag{1.9}$$

Notice that the integration in this integral is performed only over the strip $\{\zeta;\ 0 < \phi < 1\} \subset\subset \omega \setminus K$. Hence for z in a neighborhood of K we can differentiate the integral and obtain

$$f^{(m)}(z) = -\frac{m!}{\pi}\int\frac{\partial\phi}{\partial\bar\zeta}\frac{f(\zeta)d\lambda(\zeta)}{(\zeta - z)^{m+1}}, \quad z \in K.$$

In this integral, $|\zeta - z| \geq \delta$, and therefore we get the estimate

$$\sup_K |f^{(m)}| \leq \frac{m!}{\pi \delta^{m+1}} \sup \left| \frac{\partial \phi}{\partial \bar{\zeta}} \right| \int_\omega |f(\zeta)| d\lambda(\zeta).$$

□

The formula (1.9) is a variant of Cauchy's formula where the curve is replaced by a strip. The usual Cauchy formula cannot be used in the preceding proof (since we want L^1-estimates), nor in the next one (since it deals with functions only defined a.e.). However, even in some other situations it is more convenient to use (1.9) rather than the usual Cauchy formula, as will be apparent in what follows.

1.9 Proposition (Weyl's Lemma). *Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\partial f/\partial \bar{z} = 0$ weakly, i.e.,*

$$\int f \partial \phi / \partial \bar{z} = 0, \qquad \phi \in C_0^\infty(\Omega). \tag{1.10}$$

Then there is a $g \in A(\Omega)$ such that $f = g$ a.e.

Thus, in particular, if $f \in C^0(\Omega)$ and (1.10) holds, then f is analytic. An analogous result is also true (with essentially the same proof) for $f \in \mathcal{D}'(\Omega)$ (the space of distributions on Ω). Clearly, any $f \in A(\Omega)$ satisfies (1.10).

Proof. Take $\omega \subset\subset \Omega$ and $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ in a neighborhood of $\overline{\omega}$, and let

$$g(z) = -\frac{1}{\pi} \int \frac{(\partial \phi / \partial \bar{\zeta}) f(\zeta)}{\zeta - z} d\lambda(\zeta), \qquad z \in \omega.$$

If f is analytic, then (1.9) says that $g = f$ in ω; we are going to show that (1.10) actually implies that $f = g$ a.e. in ω. Since $g(z)$ is analytic in ω and $\omega \subset\subset \Omega$ is arbitrary, our proof is then complete. To do this, take $\psi \in C_0^\infty(\omega)$. By Fubini's theorem

$$\int g(z)\psi(z) = -\int_\zeta \frac{\partial \phi}{\partial \bar{\zeta}}(\zeta) \left(-\frac{1}{\pi} \int_z \frac{\psi(z)}{z - \zeta} \right) f(\zeta)$$
$$= -\int_\zeta \frac{\partial}{\partial \bar{\zeta}} \left(\phi(\zeta) \left(-\frac{1}{\pi} \int_z \frac{\psi(z)}{z - \zeta} \right) \right) f(\zeta) + \int_\zeta \phi \frac{\partial}{\partial \bar{\zeta}} \left(-\frac{1}{\pi} \int_z \frac{\psi(z)}{z - \zeta} \right) f(\zeta).$$

The first of these integrals vanishes by the assumption on f, since

$$\phi \left(-\frac{1}{\pi} \int_z \frac{\psi(z)}{z - \zeta} \right)$$

is in $C_0^\infty(\Omega)$ (why?). According to (1.7), the second integral is

$$= \int \phi(\zeta)\psi(\zeta)f(\zeta) = \int \psi(\zeta) f(\zeta)$$