Stabilities in Dynamical Systems

Changming Ding   Yuming Chu

（动力系统中的稳定性）

Science Press
Beijing

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SCIENCE PRESS
Beijing   New York

Responsible Editor: Wang Liping

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

ISBN 978-7-03-022596-2 (Science Press, Beijing)

Preface

Stability is a classical topic in the study of dynamical systems. Since the time of Lyapunov, mathematicians have proposed many different definitions of stability to investigate basic properties of orbits of dynamical systems. This book does not contain all these important results in the literature, but focuses on three special topics, i.e., chain stability, Zhukovskij stability and intertwined basins.

We begin with basic notions that are necessary to describe the dynamical behaviors, and also reach to the most recent achievements, for example, the chain stability is first proposed in 2008[9]. In this book, we are mainly interested in the geometric or topological aspects of the orbits or solutions more than an explicit formula for an orbit. Also, this book is meant to be a graduate textbook and not just only a monograph on the subject.

This book contains four chapters. All the definitions, theorems and formulae are independently numbered by chapter, for example, Theorem 2.8 in Chapter II means the eighth theorem in Section 2 of the same chapter. Chapter I gives a detailed discussion about basic concepts in dynamical systems, in particular, it deals with different recursion motions and their relations, for example, Auslander recurrence and Chain recurrence. Also, it includes some specific prerequisites for later discussions. Chapters II and III are the central part of this book, they treat two important ideas: chain stability and Zhukovskij stability. Most results in these two chapters are published recently in mathematical journals. In Chapter IV, we consider an interesting dynamical phenomenon: intertwined basins of attraction for continuous flow. Obviously, it demands further researches.

We were supported by the National Science Foundation of China and several grants from our universities.

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Contents

Chapter I

Basic Definitions and Properties

In this chapter, we introduce some important concepts in dynamical systems and discuss their basic properties, which are often used in the subsequent chapters. The general information on continuous dynamical systems or flows may be found in [24] and [4].

I.1 Some Elementary Concepts

Let X be a metric space with metric d, i.e., X is an arbitrary set of elements and there is a nonnegative real-valued function $d(p, q)$ of p and q in X, called the distance function and satisfying the following three conditions: (i) $d(p, q) = 0$ if and only if $p = q$, (ii) $d(p, q) = d(q, p)$, and (iii) $d(p, q) \leqslant d(p, s) + d(s, q)$ for arbitrary p, q, s in X. Throughout the book, we let $B_\delta(x) = \{y \in X | d(x, y) < \delta\}$ be the open ball with center x

and radius $\delta > 0$, and let $\overline{B}_\delta(x) = \{y \in X | d(x,y) \leqslant \delta\}$ be the closed ball. For $x \in X$, let $H_r(x) = \{y \in X | d(x,y) = r\}$ be the sphere centered at x. In addition, for $p \in X$ and $A \subset X$, let $d(p, A) = \inf\{d(p,z) | z \in A\}$, then the r-neighborhood of A is denoted by $N_r(A) = \{z \in X | d(z, A) < r\}$ for $r > 0$. Similarly, we define $H_r(A) = \{z \in X | d(z, A) = r\}$ for $r > 0$. For $A \subset X$, $\overline{A}$, ∂A and $\mathrm{Int}A$ denote the closure, the boundary and the interior of A, respectively. Finally, $\mathbf{R}$ and $\mathbf{R}^+$ ($\mathbf{R}^-$) denote the reals and non-negative (non-positive) reals, respectively.

Let X be a set, a *relation* F on X is a subset of $X \times X$. The inverse of F, denoted by F^{-1}, is obtained by reversing each of the pairs belonging to F, i.e., $F^{-1} = \{(x,y) | (y,x) \in F\}$. The composite of two relations F and G is the relation defined by the formula: $F \circ G = \{(x,z) \in X \times X | \exists y \in X \text{ such that } (x,y) \in F \text{ and } (y,z) \in G\}$. A relation F is called a *closed relation* if it is a closed subset in $X \times X$. A relation F is called *transitive* if $(x,y) \in F$ and $(y,z) \in F$ imply $(x,z) \in F$, or equivalently, if $F \circ F \subset F$. Note that a relation F on X can be thought of as a map from X to the power set of X associating to each $x \in X$ a subset $F(x) = \{y | (x,y) \in F\}$ of X.

If X is a metric space, let 2^X be the set of all nonempty subsets of X, and let $\mathcal{H}(X) = \{A \subset X | A$ is nonempty and closed in $X\}$. The *Hausdorff metric* for $\mathcal{H}(X)$ induced by d, which is denoted by h_d, is defined as follows: for any A, $B \in \mathcal{H}(X)$, $h_d(A, B) = \inf\{r > 0 | A \subset N_r(B) \text{ and } B \subset N_r(A)\}$. Note that if X is a bounded metric space, then h_d is a metric. Moreover, if X is compact, then $(\mathcal{H}(X), h_d)$ is also a compact metric space (see [22]). $(\mathcal{H}(X), h_d)$ is said to be the *hyperspace* of (X, d).

Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of $(X, \mathcal{T})$, where $\mathcal{T}$ is the topology induced by d. We define $\limsup A_i$ and $\liminf A_i$ as follows:

$$\limsup A_i = \{x \in X | \text{for each } U \in \mathcal{T} \text{ such that } x \in, U \cap A_i \neq \varnothing \text{ for infinitely many } i\},$$

$$\liminf A_i = \{x \in X | \text{for each } U \in \mathcal{T} \text{ such that } x \in U, U \cap A_i \neq \varnothing \text{ for all but finitely many } i\}.$$

Observe that these two limit sets defined above are tightly related to the topology $\mathcal{T}$ of the space X. Clearly, $\liminf A_i \subset \limsup A_i$. If $\limsup A_i = \liminf A_i = A$, we write $\lim A_i = A$. Indeed, if X is compact and $\{A_i\}_{i=1}^{\infty} \subset \mathcal{H}(X)$, then $\lim A_i$ is just the limit point of $\{A_i\}_{i=1}^{\infty}$ in the metric space $\mathcal{H}(X)$ in sense of Hausdorff metric.

Definition 1.1 A dynamical system or continuous flow on X is a map $\pi : X \times \mathbf{R} \to X$ satisfying

(a) $\pi(x, 0) = x$ for each $x \in X$ (identity axiom),

(b) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for $x \in X$ and t, $s \in \mathbf{R}$

(group axiom),

(c) π is continuous (continuity axiom).

When a dynamical system is given on X, the space X and the map π are respectively called the *phase space* and the *phase map*. Here, t is often considered as a time parameter. In the sequel, unless otherwise stated, a dynamical system on X is always assumed given.

For brevity, we always write $x \cdot t$ in place of $\pi(x,t)$, axioms (a) and (b) then read $x{\cdot}0 = x$ and $(x{\cdot}t){\cdot}s = x{\cdot}(t{+}s)$. Similarly, we let $A \cdot J = \{x \cdot t | x \in A, t \in J\}$ for $A \subset X$ and $J \subset \mathbf{R}$. For example, $x \cdot \mathbf{R} = \{x \cdot t | t \in \mathbf{R}\}$ and $x \cdot \mathbf{R}^+ = \{x \cdot t | t \geqslant 0\}$ are the orbit and the positive semi-orbit respectively of a point $x \in X$. A set Y in X is called positively (negatively) invariant under π if $Y \cdot \mathbf{R}^+ \subset Y$ ($Y \cdot \mathbf{R}^- \subset Y$). A set Y is said to be invariant provided $Y \cdot \mathbf{R} = Y$. Thus, an invariant set is a set consisting of entire orbits, and conversely.

Remark The closure, the boundary, the interior and the complement of an invariant set are also invariant sets.

For a point p in X, there are three possible orbits as follows.

(1) $p \cdot t = p$ for all $t \in \mathbf{R}$. In this case the point p is called a rest (critical, equilibrium, or stationary) point, its orbit is a singular point set.

(2) $p \cdot \tau = p$ for a $\tau \neq 0$ and p is not a rest point. In this

case, p is said to be periodic and τ is called a period of p, now its orbit is a simple closed curve.

(3) $p \cdot t \neq p$ for all $t \in \mathbf{R}$. Thus, the orbit $p \cdot \mathbf{R}$ is a one-to-one continuous image of the real line.

The following important property is obtained directly from the continuity axiom.

Continuous Dependence on Initial Conditions: For any point $x \in X$, any positive number T (arbitrarily large) and any $\varepsilon > 0$ (arbitrarily small), there can be found a $\delta > 0$ such that if $d(x, y) < \delta$ then there holds the inequality $d(x \cdot t, y \cdot t) < \varepsilon$ for $t \in [-T, T]$.

Proof Suppose on the contrary, there exist sequences of points $\{x_n\}$, $x_n \to x$ $(n \to \infty)$ and numbers $\{t_n\}$, $t_n \in [-T, T]$, such that $d(x \cdot t_n, x_n \cdot t_n) \geqslant \lambda > 0$. Since $\{t_n\}$ is bounded, it has a convergent subsequence. Without loss of generality, we assume that $t_n \to \tau \in [-T, T]$. Now, it follows from the continuity axiom that

$$d(x \cdot t_n, x_n \cdot t_n) \leqslant d(x \cdot t_n, x \cdot \tau) + d(x \cdot \tau, x_n \cdot t_n) \to 0 \quad (n \to \infty),$$

it leads to $\lambda \leqslant 0$, a contradiction. □

Definition 1.2 For a subset S in X and $T > 0$, if the set $\Sigma = S \cdot (-T, T)$ is open in X, and to each point $x \in \Sigma$ there corresponds a unique number t_x such that $x \cdot t_x \in S$ and $|t_x| < T$, then we call Σ a tube of time length $2T$ with section (or transversal) S.

Theorem 1.3 If p is not a rest point of a dynamical system, then there exists a tube containing p.

Proof Since p is not a rest point, there exists a $\tau > 0$ such that $d(p, p\cdot\tau) > 0$. Let $\phi(x,t) = \int_t^{t+\tau} d(p, x\cdot\theta)d\theta$, it is easy to see that $\phi(x,t)$ is continuous in (x,t) and has the continuous partial derivative $\phi_t(x,t) = d(p, x\cdot(t+\tau)) - d(p, x\cdot t)$. Also,

$$\begin{aligned}\phi(x, t_1+t_2) &= \int_{t_1+t_2}^{t_1+t_2+\tau} d(p, x\cdot\theta)d\theta \\ &= \int_{t_2}^{t_2+\tau} d(p, x\cdot(\theta+t_1))d\theta \\ &= \int_{t_2}^{t_2+\tau} d(p, (x\cdot t_1)\cdot\theta)d\theta \\ &= \phi(x\cdot t_1, t_2).\end{aligned}$$

Since $\phi_t(p,0) = d(p, p\cdot\tau) > 0$, there is an $\varepsilon > 0$ such that $\phi_t(x,0) > 0$ for $x \in B_\varepsilon(p)$. Now, according to the continuous dependence on initial conditions, define $\tau_0 > 0$ such that $p\cdot[-3\tau_0, 3\tau_0] \subset B_\varepsilon(p)$, which also implies that $\phi(p,\tau_0) > \phi(p,0) > \phi(p,-\tau_0)$. Next choose $\eta > 0$ such that $\overline{B}_\eta(p\cdot\tau_0) \cup \overline{B}_\eta(p\cdot(-\tau_0)) \subset B_\varepsilon(p)$, meanwhile, for $x \in B_\eta(p\cdot\tau_0)$ we have $\phi(x,0) > \phi(p,0)$ and for $x \in B_\eta(p\cdot(-\tau_0))$ we have $\phi(x,0) < \phi(p,0)$. Finally, let $\delta > 0$ such that $\overline{B}_\delta(p)\cdot[-3\tau_0, 3\tau_0] \subset B_\varepsilon(p)$, $\overline{B}_\delta(p)\cdot\tau_0 \subset B_\eta(p\cdot\tau_0)$ and $\overline{B}_\delta(p)\cdot(-\tau_0) \subset B_\eta(p\cdot(-\tau_0))$. We assert that if $x \in \overline{B}_\delta(p)$, then there is exactly one $t(x)$ satisfying $|t(x)| < \tau_0$ and $\phi(x, t(x)) = \phi(p, 0)$. Actually, it follows from the fact that $\phi(x,t) = \phi(x \cdot t, 0)$ is a strictly increasing function of t and $\phi(p,\tau_0) > \phi(p,0) > \phi(p,-\tau_0)$. Let $\Sigma = B_\delta(p) \cdot (-\tau_0, \tau_0)$ and $L = \{x \in \Sigma | \phi(x,0) = \phi(p,0)\}$. We shall prove that L is a local section of the tube Σ, i.e, for each $x \in \Sigma$ there is a unique $t(x)$ with $|t(x)| < 2\tau_0$ such that $x \cdot t(x) \in L$. In fact, by the definition of Σ, for any $x \in \Sigma$ there is a τ_1 with $|\tau_1| < \tau_0$ such that $x_1 = x \cdot \tau_1 \in B_\delta(p)$, and then for $x_1 \in B_\delta(p)$ there is a τ_2 with $|\tau_2| < \tau_0$ such that $x_1 \cdot \tau_2 \in L$. Thus $x \cdot (\tau_1 + \tau_2) = x \cdot t(x) \in L$, where $t(x) = \tau_1 + \tau_2$ satisfying $|t(x)| \leqslant |\tau_1| + |\tau_2| < 2\tau_0$. In order to prove the uniqueness of $t(x)$, we let t' and t'' in $[-\tau_0, \tau_0]$ such that $x \cdot t' \in L$ and $x \cdot t'' \in L$, and also suppose $x_1 = x \cdot \tau_1 \in B_\delta(p)$ with $|\tau_1| < \tau_0$. Hence, $\phi(x_1, t' - \tau_1) = \phi(x, t') = \phi(x \cdot t', 0)$ and $\phi(x_1, t'' - \tau_1) = \phi(x, t'') = \phi(x \cdot t'', 0)$, which implies that $\phi(x_1, t' - \tau_1) = \phi(x_1, t'' - \tau_1) = \phi(p, 0)$. Note that $|t' - \tau_1| \leqslant 3\tau_0$, $|t'' - \tau_1| \leqslant 3\tau_0$ and $\phi_t(x_1, t) > 0$ for $|t| < 3\tau_0$, which implies that $\phi(x,t)$ is strictly increasing for $|t| \leqslant 3\tau_0$. Thus, $t' - \tau_1 = t'' - \tau_1$, or $t' = t''$. So we are done. □

Remark If X is a locally compact metric space, we may assume that the function $t(x)$ in Theorem 1.3 is continuous in Σ. In fact, now we can restrict $\delta > 0$ in the above proof to ensure that $\overline{B}_\delta(p)$ is compact, thus L is locally compact.

Consider a sequence $\{x_n\}$ in Σ and $x_n \to x \in \Sigma$. Note that $\{x_n \cdot t(x_n)\}$ lies in $L \subset \overline{B}_\delta(p)$, without loss of generality, we suppose that it is convergent to x' in $\overline{L}$. Also, since $\{t(x_n)\}$ is in $[-\tau_0, \tau_0]$, we assume that $t(x_n) \to t'$. Hence, we have $x' = x \cdot t'$. From $|t'| \leqslant \tau_0$, it follows that $t' = t(x)$ by uniqueness, i.e., $t(x_n) \to t(x)$. Then $t(x)$ is continuous.

I.2 Recurrent Orbits

In this section, we first introduce variant limit sets, which play a central role in the study of asymptotic behaviors of a dynamical system. Next, we define several recursion concepts and present their characterizations.

Definition 2.1 The positive limit set or omega limit set of x in X is the set $\omega(x) = \{y \in X |$ there is a sequence $\{t_n\} \subset \mathbf{R}^+$ such that $t_n \to +\infty$ and $x \cdot t_n \to y\}$. The first positive prolongational limit set and first positive prolongational set are defined, respectively, by $J^+(x) = \{y \in X|$ there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^+$ such that $x_n \to x$, $t_n \to +\infty$ and $x_n \cdot t_n \to y\}$ and $D^+(x) = \{y \in X|$ there are two sequences $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbf{R}^+$ such that $x_n \to x$ and $x_n \cdot t_n \to y\}$.

By reversing the direction of time, we obtain definitions of the negative cases $\alpha(x)$, $J^-(x)$ and $D^-(x)$. In the sequel, we will focus on the discussions about $\omega(x)$, $J^+(x)$ and $D^+(x)$, however, similar results hold for $\alpha(x)$, $J^-(x)$ and $D^-(x)$.

In this book, if no confusion, $J^+(x)$ and $D^+(x)$ are often called prolongational limit set and prolongational set for short, respectively.

Theorem 2.2 For any $x \in X$, the following are true.

(1) $\omega(x)$ and $J^+(x)$ are closed and invariant, $D^+(x)$ is closed and positively invariant;

(2) $\overline{x \cdot \mathbf{R}^+} = x \cdot \mathbf{R}^+ \cup \omega(x)$ and $D^+(x) = x \cdot \mathbf{R}^+ \cup J^+(x)$;

(3) $\omega(x \cdot t) = \omega(x)$ and $J^+(x \cdot t) = J^+(x)$ hold for each $t \in \mathbf{R}$.

Proof (1) We first show that $\omega(x)$ is closed and invariant. Let $\{y_n\}$ be a sequence in $\omega(x)$ with $y_n \to y$. Then, for each $n \geqslant 1$, there is a sequence $\{t_k^n\}$ in $\mathbf{R}^+$ with $t_k^n \to +\infty$ and $x \cdot t_k^n \to y_n$ as $k \to +\infty$. Choose $k_n \geqslant n$ such that $d(y_n, x \cdot t_{k_n}^n) < \dfrac{1}{n}$ for every $n \geqslant 1$, and let $t_n = t_{k_n}^n$. Thus $t_n \to +\infty$ and we claim that $x \cdot t_n \to y$. To see this, observe that $d(y, x \cdot t_n) \leqslant d(y, y_n) + d(y_n, x \cdot t_n) \leqslant d(y, y_n) + \dfrac{1}{n}$. So we conclude that $d(y, x \cdot t_n) \to 0$ as $n \to +\infty$, and then $y \in \omega(x)$. Consequently, $\omega(x)$ is closed. To see that $\omega(x)$ is invariant, let $y \in \omega(x)$ and $t \in \mathbf{R}$ be arbitrary. There is a sequence $\{t_n\}$ in $\mathbf{R}$ with $t_n \to +\infty$ and $x \cdot t_n \to y$. From the continuity axiom, it follows $x \cdot (t_n + t) \to y \cdot t$. Since $t_n + t \to +\infty$, we have $y \cdot t \in \omega(x)$, and $\omega(x)$ is invariant. Next, to show that $J^+(x)$ is closed and invariant, let $\{y_n\}$ be a sequence in $J^+(x)$ with $y_n \to y$. For each positive integer n, there are sequences $\{x_k^n\}$ in X and $\{t_k^n\}$ in $\mathbf{R}^+$ with $x_k^n \to x$, $t_k^n \to +\infty$ and $x_k^n \cdot t_k^n \to y_n$. Choose $k_n \geqslant n$ such that $d(x_{k_n}^n, x) < \frac{1}{n}$ and $d(y_n, x_{k_n}^n \cdot t_{k_n}^n) < \frac{1}{n}$ for every $n \geqslant 1$. Let $x_n = x_{k_n}^n$ and $t_n = t_{k_n}^n$. Then, $x_n \to x$, $t_n \to +\infty$ and $x_n \cdot t_n \to y$ as $n \to +\infty$. Hence $J^+(x)$ is closed. To see that $J^+(x)$ is invariant, let $y \in J^+(x)$ and $\tau \in \mathbf{R}$ be arbitrary. There are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^+$ such that $t_n \to +\infty$, $x_n \to x$ and $x_n \cdot t_n \to y$. Clearly, $t_n + \tau \to +\infty$ and $x_n \cdot (t_n + \tau) \to y \cdot \tau$. Since $x_n \to x$, we have $y \cdot \tau \in J^+(x)$ and $J^+(x)$ is invariant. Finally, it is easy to see, from $D^+(x) = x \cdot \mathbf{R}^+ \cup J^+(x)$ in (2), that $D^+(x)$ is closed and positively invariant.

(2) By the definition of $\omega(x)$, we have $\overline{x \cdot \mathbf{R}^+} \supset x \cdot \mathbf{R}^+ \cup \omega(x)$. To see that $\overline{x \cdot \mathbf{R}^+} \subset x \cdot \mathbf{R}^+ \cup \omega(x)$, let $y \in \overline{x \cdot \mathbf{R}^+}$. Then there is a sequence $\{y_n\}$ in $x \cdot \mathbf{R}^+$ such that $y_n \to y$. Now $y_n = x \cdot t_n$ for a $t_n \in \mathbf{R}^+$. Either the sequence $\{t_n\}$ has the property that $t_n \to +\infty$, in which case $y \in \omega(x)$, or there is a subsequence $t_{n_k} \to t \in \mathbf{R}^+$. But then $x \cdot t_{n_k} \to x \cdot t \in x \cdot \mathbf{R}^+$, and since also $x \cdot t_{n_k} \to y$ we have $y = x \cdot t \in x \cdot \mathbf{R}^+$. Thus $\overline{x \cdot \mathbf{R}^+} \subset x \cdot \mathbf{R}^+ \cup \omega(x)$. Next, we prove that $D^+(x) = x \cdot \mathbf{R}^+ \cup J^+(x)$. By the definition of $D^+(x)$, it is easy to see