# Point Set Topology

S. A. GAAL

# POINT SET TOPOLOGY

BY

STEVEN A. GAAL

*Department of Mathematics*
*University of Minnesota*
*Minneapolis, Minnesota*

1964 ACADEMIC PRESS New York and London

ACADEMIC PRESS INC.
111 Fifth Avenue, New York, New York 10003

*United Kingdom Edition published by*
ACADEMIC PRESS INC. (LONDON) LTD.
Berkeley Square House, London W.1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 64-21667

PRINTED IN THE UNITED STATES OF AMERICA

# Preface

This book for beginning graduate and advanced undergraduate mathematics students presents point set topology not only as an end in itself, but also as a related discipline to the proper understanding of various branches of analysis and geometry. It starts with the basic concepts of set theory and topological spaces and ends with the beginning of functional analysis. The text and nearly all of the exercises presuppose knowledge of only those concepts defined herein, so that the book is self-contained to accommodate those who wish to study topology on their own. Moreover, it includes additional material and literature which make it valuable as a reference work.

The book contains enough material for a one-year course, and I have found it accessible to juniors majoring in mathematics. By omitting carefully chosen sections it is possible, but not easy, to cover most of the book in a one-semester course. When giving a year's course on the foundations of analysis for graduate students, I have been able to include some additional material, such as differentiable manifolds or abstract harmonic analysis and fixed point theorems.

The first chapter contains the fundamental notions associated with a general topological space, and a systematic discussion of the various practical methods used to define topological spaces. The second and third chapters deal with those additional properties that give a general topological space more resemblance to the primitive, intuitive concept associated with the concept of space. The basic properties of functions defined on topological spaces are collected in the fourth chapter, although some concepts, such as continuous maps and homeomorphisms, were introduced earlier for the sake of clarity. The last chapter contains an exposition of the theory of topological convergence using filters and nets, which is applied to problems of compactness, completion, and compactification. The exercises vary in difficulty, and some provide additional insights or new results. The principal theorems are all included as part of the text. The remarks at the end of each chapter contain pertinent comments which did not seem suitable for inclusion elsewhere. While it is hoped that they may give the reader some historical perspective, they do reflect the author's personal thoughts, and no attempt has been made at complete coverage.

The bulk of the material presented here was developed through lectures at Cornell University and the University of Minnesota. I am very grateful to Professor R. J. Walker and to Miss Madelyn Keady of Cornell University and to my wife for their encouragement and help with the preliminary publication. A great deal of additional work and support is needed to make a first draft develop into a book. I want to thank Professor J. B. Rosser for his kind interest in my project and for giving me a helping hand. In Minnesota I was very fortunate to meet Mr. Glenn Schober, who read the entire manuscript and made many improvements. He also read and corrected the proofs and thus helped eliminate many errors that I had overlooked.

I also want to express my appreciation for the support given to me by the various Federal agencies at several stages of the project. Finally, my sincere thanks to Academic Press for their careful work.

STEVEN A. GAAL

*June, 1964*

# Notation

| | |
|---|---|
| $\mathscr{B}$ | Base for open sets; 33 |
| $\mathscr{B}$ | Filter base; 43 |
| $\mathscr{C}$ | Family of closed sets; 22 |
| $\mathfrak{C}$ | Complex number field; 1 |
| $\mathscr{C}(x)$ | Filter base of closed neighborhoods of $x$; 43 |
| $E_n$ | Euclidean space of dimension $n$; 40 |
| $\mathscr{F}$ | Filter; 43 |
| $I$ | Diagonal; 6, 46 |
| $N_x$ | Neighborhood of $x$; 42 |
| $\mathscr{N}(x)$ | Neighborhood filter of $x$; 43 |
| $O$ | Open set; 21 |
| $\mathscr{O}$ | Family of open sets; 21 |
| $\mathscr{O}(x)$ | Filter base of open neighborhoods of $x$; 43 |
| $\mathscr{P}(X)$ | Power set of $X$; 6, 7 |
| $R$ | Field of rationals; 1 |
| $\mathfrak{R}$ | Field of real numbers; 1 |
| $\mathscr{S}$ | Subbase for open sets; 34 |
| $S_\epsilon[x]$ | $\epsilon$-Ball with center at $x$; 38 |
| $\mathscr{T}$ | Topology; 21 |
| $\mathscr{T}_+$ | Right half-open interval topology; 37 |
| $\mathscr{T}_-$ | Left half-open interval topology; 37 |
| $\mathscr{T}_R$ | Topology determined by the saturated open sets relative to $R$; 73 |
| $\mathscr{T}/R$ | Quotient topology; 71 |
| $U$ | Uniformity; 46 |
| $U^{-1}$ | Inverse uniformity; 46 |
| $U[x]$ | Uniformity $U$ evaluated at $x$; 48 |
| $U[A]$ | Uniformity $U$ evaluated on the set $A$; 48 |
| $U_Y$ | Trace of uniformity $U$ on $Y \times Y$; 57 |
| $\mathscr{U}$ | Uniform structure; 45 |
| $\mathscr{U}_B$ | Structure base; 46 |
| $\mathscr{U}_S$ | Subbase for a uniform structure; 47 |
| $\mathscr{U}_Y$ | Trace of uniform structure $\mathscr{U}$ on set $Y$; 57 |
| $\{a\}$ | Set consisting of single element $a$; 2 |
| $\{a, b\}$ | Nonordered pair of $a$ and $b$; 2 |
| $\{A_n\}$ $(n = 1, 2, \ldots)$ | Countable or denumerable family; 8 |
| $(A_n)$ $(n = 1, 2, \ldots)$ | Sequence, i.e. countable family with a fixed enumeration; 8 |
| $\{A_i\}$ $(i \in I)$ | Family of objects with index set $I$; 13 |
| $(A_i)$ $(i \in I)$ | Indexed family of objects with partially ordered index set $I$; 14 |
| $(O_d)$ $(d \in D)$ | Scale of open sets; 109 |
| $A^e$ | Exterior of $A$; 24 |
| $A^b$ | Boundary of $A$; 25 |
| $A^i$ | Interior of $A$; 24 |
| $\bar{A}$ | Closure of $A$; 25 |
| $A'$ | Derived set of $A$; 26 |
| $-B$ | Complement of $B$; 3 |
| $cB$ | Complement of $B$; 3 |
| $c_A B$ | Complement of $B$ relative to $A$; 3 |
| $A \underset{r}{\cup} B$ | Regular union of $A$ and $B$; 27 |
| $A \underset{r}{\cap} B$ | Regular intersection of $A$ and $B$; 27 |
| $A - B$ | Relative complement; 3 |
| $A \setminus B$ | Complement of $B$ relative to $A$; 3 |
| $A \triangle B$ | Symmetric difference of $A$ and $B$; 4 |
| $A \circ B$ | Composition of $A$ and $B$; 46, 176 |
| $S^b_Y$ | Boundary of $S$ relative to subspace $Y$; 56 |
| $(a, b)$ | Ordered pair of $a$ and $b$, or open interval with end points $a$ and $b$; 4, 5, 35 |

| | |
|---|---|
| $[a, b]$ | Closed interval with end points $a$ and $b$; 35 |
| $(a, b]$ | Left half-open interval; 35 |
| $[a, b)$ | Right half-open interval; 35 |
| $(a, +\infty)$ | Improper interval with left end point $a$; 35 |
| $(-\infty, b)$ | Improper interval with right end point $b$; 35 |
| $(a_1, \dots, a_n)$ | Ordered $n$-tuple; 5 |
| $(A \times B)$ | Product of the ordered pair $(A, B)$ of sets $A$, $B$; generally written as $A \times B$ when it causes no confusion; 5 |
| $(A_1 \times \dots \times A_n)$ | Product of the ordered $n$-tuple $(A_1, \dots, A_n)$ of sets $A_1, \dots, A_n$; 5 |
| $A_1 \times \dots \times A_n$ | Cartesian product on the natural numbers; 14 |
| $\prod X_s$ | Product of the sets $X_s$ or the product of the topological spaces $X_s$ with the product topology; 59 |
| $\pi_s(A)$ | Projection of $A$ into the $s$th factor; 59 |
| $Z(s, A_s)$ | Cylinder with base $A_s$ in the $s$th factor; 59 |
| $x^*$ | Star of a point $x$; 143 |
| $d(X)$ | Diameter of the metric space $X$; 133 |
| $d(a, b)$ | Metric function; distance of $a$ and $b$; 38 |
| $d(A, B)$ | Distance of sets $A$ and $B$ in the metric space having metric $d$; 141 |
| $A \leqslant B$ | Cardinality of $A$ is at most as large as that of $B$; 12 |
| $A < B$ | The set $A$ is of smaller cardinality than $B$; 12 |
| $\mathcal{T}_1 \leqslant \mathcal{T}_2$ | The topology $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$; 22 |
| $\mathcal{T}_1 < \mathcal{T}_2$ | The topology $\mathcal{T}_1$ is strictly coarser than $\mathcal{T}_2$; 22 |
| $\mathcal{U}_1 \leqslant \mathcal{U}_2$ | The uniform structure $\mathcal{U}_1$ is coarser than $\mathcal{U}_2$; 53 |
| $\mathcal{U}_1 < \mathcal{U}_2$ | The uniform structure $\mathcal{U}_1$ is strictly coarser than $\mathcal{U}_2$; 53 |
| $\mathcal{T}_x \sim \mathcal{T}_y$ | The topological spaces $X$ and $Y$ are homeomorphic; 22 |
| $a\, R\, b$ | $a$ is related to $b$ by the relation $R$; 5 |
| $A \sim B$ | $A$ and $B$ are equivalent sets; 7 |
| $R_2/R_1$ | Quotient of the equivalence relations $R_1$ and $R_2$; 72 |
| $f: A \to B$ | Function on $A$ with values in $B$; occasionally function from $A$ in $B$; 6, 175 |
| $f \circ g$ | Composition of functions $f$ and $g$; 177 |
| $f \cup g$ | Maximum of $f$ and $g$; 185 |
| $f \cap g$ | Minimum of $f$ and $g$; 185 |
| $f(\mathcal{T})$ | Direct image of the topology $\mathcal{T}$ under the map $f$; 70 |
| $f^{-1}(A)$ | Inverse image of the set $A$ under $f$; 68 |
| $f^{-1}(\mathcal{T})$ | Inverse image of the topology $\mathcal{T}$ under $f$; 68 |
| $F \circ G$ | Composition of graphs; 176 |
| $\mathrm{glb}\{\mathcal{F}_\alpha\}$ | Greatest lower bound of the filters $\mathcal{F}_\alpha$; 262 |
| $\mathrm{glb}\{\mathcal{T}_i\}$ | Greatest lower bound of the topologies $\mathcal{T}_i$; 34 |
| $\mathrm{lub}\{\mathcal{F}_\alpha\}$ | Least upper bound of the filters $\mathcal{F}_\alpha$ provided it exists; 262 |
| $\mathrm{lub}\{\mathcal{T}_i\}$ | Least upper bound of the topologies $\mathcal{T}_i$; 34 |
| $\mathrm{lub}\{\mathcal{U}_i\}$ | Least upper bound of the uniform structures $\mathcal{U}_i$; 53 |
| $\mathrm{adh}\, \mathcal{B}$ | Adherence of the filter base $\mathcal{B}$; 259 |
| $\mathrm{adh}\, \mathcal{F}$ | Adherence of the filter $\mathcal{F}$; 259 |
| $\lim \mathcal{B}$ | Limit of the filter base $\mathcal{B}$; 260 |
| $\lim \mathcal{F}$ | Limit of the filter $\mathcal{F}$; 260 |
| $\mathrm{adh}(x_d)$ | Adherence of the net $(x_d)$; 261 |
| $\lim(x_d)$ | Limit of the net $(x_d)$; 262 |
| $\varnothing$ | Empty set, void set; 1 |
| $\cup$ | Union, maximum; 3, 185 |
| $\cap$ | Intersection, minimum; 3, 185 |
| $\bigcup$ | Union; 3 |
| $\bigcap$ | Intersection; 3 |
| $\prod$ | Cartesian product, product; 14 |
| $\in$ | Membership relation; 1, 2 |
| $\notin$ | Negation of the membership relation; 1 |
| $\subset$ | Proper inclusion; 2 |
| $\subseteq$ | Improper inclusion, subset of ...; 2 |
| $\not\subset$ | Negation of proper inclusion; 2 |
| $\not\subseteq$ | Negation of improper inclusion; 2 |

# Contents

# Introduction to Set Theory

## 1. Elementary Operations on Sets

At the time of this writing the fundamental system in mathematics from which all others are constructed by logical reasonings is the theory of sets. This is a fairly recent discovery, for the concept of an abstract set had been isolated only at the end of the last century by Georg Cantor. His results were published in two famous papers dated 1895 and 1897.

The purpose of this introductory chapter is to help the reader in forming an intuitive concept of an abstract set. Cantor's first memoir begins with the following sentences: "By a set $M$ we understand any collection into a whole of definite and separate objects $m$ of our intuition or our thought. These objects are called the elements of $M$. In symbols we express this as follows: $M = \{m\}$." Needless to say, this is far from a precise definition and it would be very foolish to build the whole of mathematics on such shaky foundations. However, these are just the opening notes of a magnificent theory and one hardly could begin differently.

Sets will be denoted by the symbols $a, b, c, \ldots$; $A, B, C, \ldots$; $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ or $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$. It is likely that the reader knows a great variety of things which he will correctly recognize as being sets: Collections of common objects, aggregates of people, finite families of natural numbers, the collection of all natural numbers $\{1, 2, 3, \ldots\}$, the set of integers $\mathfrak{Z}$, the rationals $R$, the reals $\mathfrak{R}$, the complex numbers $\mathfrak{C}$, the Euclidean spaces. These are very valuable examples of sets and we suppose that the reader is familiar with these concepts, but as far as the theory is concerned they are irrelevant. Instead, one stipulates the existence of one particular set which will be called the *void set* or the *empty set* and shall always be denoted by the symbol $\emptyset$.

Some sets are elements of others. If $a$ is an element of $A$ we write $a \in A$ and if not then $a \notin A$. Throughout this book the notations will be chosen in the following manner: If a reasoning involves sets and their elements, then the symbols $A, B, C, \ldots$ and $a, b, c, \ldots$ will be used. If we deal also with sets of sets, then these will be denoted by $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ and their elements by $A, B, C, \ldots$ . On a few occasions we shall go a step further and use a set $\mathfrak{A}$, its elements $\mathscr{A}, \mathscr{B}, \ldots$, the elements $A, B, C, \ldots$ of these, and also the elements $a, b, c, d, \ldots$ of the sets $A, B, C, \ldots$ . In the abstract axiomatic theory of sets, the *membership relation* $a \in A$ is a primitive notion and it is further characterized only by the axiomatic statements in which it occurs. One of these states that if $a$ is a set then $a \notin \emptyset$. In other words, the empty set is really void.

If every element of $A$ belongs to $B$, we say that $A$ is a *subset* of $B$ and write $A \subseteq B$. The symbol $A \nsubseteq B$ will be used to denote that $A$ is not a subset of $B$. If $A \subseteq B$ but $B \nsubseteq A$, then $A$ is called a *proper subset* of $B$ and one uses the notation $A \subset B$. Two sets $A$ and $B$ are considered identical if they have the same elements, i.e., if $A \subseteq B$ and $B \subseteq A$. If this is the case we write $A = B$. Clearly, $\emptyset \subseteq A$ for every set $A$ and if $A \neq \emptyset$ then $\emptyset \subset A$.

We shall often define sets by specifying their elements. If the set is finite we simply enclose between curly brackets all the symbols which designate these elements. For instance $\{1, 2, 4, 8\}$ or {Conn., Minn., N.J.} or $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. For sets for which this notation would be too cumbersome to handle or if the set to be described is not finite, we shall use other means. For example,

$$\{n : n = k^3 \quad \text{and} \quad k = 1, 2 \ldots\}$$

denotes the set of cubes of natural numbers and

$$\{n : n = x^2 + y^2 \quad \text{and} \quad x, y \in 3\}$$

is the set of integers which can be written as the sum of two squares. In terms of this symbolism we have $A = \{a : a \in A\}$ for any set $A$.

Our intuition suggests that the families described above indeed are sets. In the strict axiomatic treatment it is considerably harder to establish the fact that we are really defining certain sets. For instance, the existence of the natural numbers is not taken for granted but is derived from the axioms. The particular axiom which is needed here states that if $a$ and $b$ are sets then there is a set whose elements are just the sets $a$ and $b$. This is called the *nonordered pair* of $a$ and $b$ and is denoted by $\{a, b\}$. By choosing $a = b$ we obtain the existence of a set whose sole element is the set $a$. It is convenient to use the simpler notation $\{a\}$ instead of $\{a, a\}$. For $a$ we can certainly choose the void set $\emptyset$ and get a new set $\{\emptyset\}$. Then $a = \{\emptyset\}$ yields $\{\{\emptyset\}\}$, the set whose only element is a set consisting of the single element $\emptyset$. By continuing in this fashion we end up with a whole class of sets:

$$\{\emptyset\}, \quad \{\{\emptyset\}\}, \ldots, \{\ldots\{\{\emptyset\}\}\ldots\}, \ldots .$$

Special names and symbols are attached to these curious sets: *One*, *two*, *three*, ... and 1, 2, 3, ...

If $\mathscr{A}$ is a set whose elements $A$ are also sets, then we let $\bigcup \mathscr{A}$ denote the set consisting of the elements of the sets $A$:

$$\bigcup \mathscr{A} = \{a : a \in A \quad \text{and} \quad A \in \mathscr{A}\}.$$

This is called the *union* of $\mathscr{A}$ or the sum of $\mathscr{A}$. Its existence is postulated in one of the axioms. If $\mathscr{A}$ is finite, e.g., if $\mathscr{A} = \{A, B\}$ or $\mathscr{A} = \{A, B, C\}$, then we shall write $A \cup B$ or $A \cup B \cup C$ instead of $\bigcup \mathscr{A}$. We notice that $A \cup B = B \cup A$ and $A \cup (B \cup C) = A \cup B \cup C = (A \cup B) \cup C$. Moreover, $A \cup A = A$ and $A \cup \varnothing = A$ for any set $A$. Similarly, we define the *intersection* of $\mathscr{A}$ as the set of common elements of the sets $A$ belonging to $\mathscr{A}$:

$$\bigcap \mathscr{A} = \{a : a \in A \quad \text{for every} \quad A \in \mathscr{A}\}.$$

In the finite case we use the notation $A \cap B$ or $A \cap B \cap C$, etc. We have $A \cap B = B \cap A$ and $A \cap (B \cap C) = A \cap B \cap C = (A \cap B) \cap C$ and also $A \cap A = A$ and $A \cap \varnothing = \varnothing$.

The operations $A \cup B$ and $A \cap B$ are meaningful for any pair of sets $A$, $B$ and yield new sets. Moreover, as we have seen, these operations follow the *commutativity* and *associativity* laws known from elementary algebra. The empty set $\varnothing$ plays the role of the *zero element*. We also have two *distributivity* properties: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

For any two sets $A$, $B$ we can define the *relative difference* $A - B$ as the set consisting of those points of $A$ which do not belong to $B$:

$$A - B = \{a : a \in A \quad \text{and} \quad a \notin B\}.$$

Thus $A - B$ is a subset of $A$ and $A - B = A$ if and only if $A$ and $B$ are *disjoint*, i.e., $A \cap B = \varnothing$. Very often $A - B$ is called the *complement of $B$ relative to $A$* and one writes $c_A B$ or $A \setminus B$ instead of $A - B$. If the set $A$ is fixed throughout some discussion it is customary to use the simpler notation $-B$ or $cB$ in place of $A - B$. This is particularly convenient when a set $A$ is given and the reasoning involves only subsets of this universe $A$.

There are two important identities involving unions, intersections, and relative complements called *de Morgan's formulas*. The simplest case concerns two sets $A$, $B$ and complementation with respect to a third fixed set $X$. In this special case de Morgan's laws are

$$c(A \cup B) = cA \cap cB \qquad \text{and} \qquad c(A \cap B) = cA \cup cB$$

or in full details

$$X-(A \cup B)=(X-A) \cap (X-B) \text{ and } X-(A \cap B)=(X-A) \cup (X-B).$$

Similarly, in the case of finitely many sets $A_1, \dots, A_n$ we have

$$c(A_1 \cup \dots \cup A_n) = cA_1 \cap \dots \cap cA_n$$

and

$$c(A_1 \cap \dots \cap A_n) = cA_1 \cup \dots \cup cA_n .$$

The general laws concern an arbitrary set $\mathscr{A}$ of sets $A$ and complementation with respect to a fixed set $X$:

$$c \bigcup \mathscr{A} = \bigcap \{cA : A \in \mathscr{A}\} \quad \text{and} \quad c \bigcap \mathscr{A} = \bigcup \{cA : A \in \mathscr{A}\}.$$

Thus the complement of the union of $\mathscr{A}$ is the intersection of the set consisting of the complements of the elements of $\mathscr{A}$ and a similar statement holds for the complement of the intersection of $\mathscr{A}$. The finite cases discussed earlier are obtained by taking as $\mathscr{A}$ the finite sets $\{A, B\}$ and $\{A_1, \dots, A_n\}$.

The subsets of a given set $X$ form a set $\mathscr{P}(X)$ which is called the *power set* of $X$. Since $\varnothing \subseteq X$ and $X \subseteq X$ we have $\varnothing, X \in \mathscr{P}(X)$ no matter what $X$ is. If $X = \varnothing$, then of course $\mathscr{P}(X) = \{\varnothing\}$. The operations $\cup$ and $\cap$ induce an interesting algebraic structure on $\mathscr{P}(X)$. As we have seen already, both of these operations are *idempotent, commutative,* and *associative.* Furthermore, they jointly obey two *distributivity laws.* The algebraic structure can be further strengthened by considering also the unary operation derived from complementation relative to $X$ and the partial ordering relation $\subseteq$ given by inclusion. Those who are familiar with the elements of abstract algebra recognize $\mathscr{P}(X)$ with this structure as a *Boolean algebra.*

The set $A \triangle B = (A-B) \cup (B-A)$ is called the *symmetric difference* of $A$ and $B$. If $A, B \in \mathscr{P}(X)$, then $A \triangle B$ being a subset of $A \cup B$ we have also $A \triangle B \in \mathscr{P}(X)$. There are several identities involving the operations $\triangle$ and $\cap$. One finds that $\mathscr{P}(X)$ is a *commutative ring* with respect to these operations which has an identity, namely $X$, and in which every element is idempotent. In other words, $\mathscr{P}(X)$ is a *Boolean ring* under the *addition* $\triangle$ and *multiplication* $\cap$. The operations $\cup$ and $\cap$ are often called "cup" and "cap" or "join" and "meet." The same terminology occurs in lattice theory.

Let $A$ and $B$ be nonvoid sets and let $a \in A$, $b \in B$. The existence of the *ordered pair* $(a, b)$ is intuitively obvious and we may also speak about the set of all these pairs $(a, b)$. This set is called the *product* of the ordered pair $(A, B)$ and it will be denoted by $(A \times B)$. Thus,

$$(A \times B) = \{(a, b) : a \in A \quad \text{and} \quad b \in B\}.$$

If $A \neq B$, then $(A \times B)$ and $(B \times A)$ are distinct sets, but if $A = B$, then these factors play symmetric roles and we have only one product which we denote by $A \times A$ or $A^2$. In axiomatic set theory the ordered pair $(a, b)$ is constructed as follows: One of the axioms which was explicitly mentioned earlier states the existence of nonordered pairs. In particular, it implies the existence of the sets $\{a\}$, $\{b\}$, and $\{a, b\}$. By the same principle we may form the nonordered pair $\{\{a\}, \{a, b\}\}$ which we call the *ordered pair* and denote by $(a, b)$. This set really has all the characteristic properties attributed to an ordered pair: It is asymmetric and the nonordered pair $\{a, b\}$ is determined by $(a, b)$, namely, it is its sum. An alternative definition of an ordered pair could be $(a, b) = \{\{a\}, \{\{b\}\}\}$. *Ordered triples* can be easily defined in terms of ordered pairs: $(a, b, c) = (a, (b, c))$. More generally we can introduce *ordered n-tuples* by using ordered $(n - 1)$-tuples as follows:

$$(a_1, \ldots, a_n) = (a_1, (a_2, \ldots, a_n)).$$

The set

$$(A_1 \times \ldots \times A_n) = \{(a_1, \ldots, a_n) : a_1 \in A_1, \ldots, a_n \in A_n\}$$

is called the *product* of the ordered $n$-tuple $(A_1, \ldots, A_n)$ of sets $A_1, \ldots, A_n$. The sets $A_1, \ldots, A_n$ are the *factors* of the product $(A_1 \times \ldots \times A_n)$. It is important to realize that the product is defined only when the distinct sets of the finite family $A_1, \ldots, A_n$ have been arranged in a definite order. If all these sets coincide, say $A_1 = \ldots = A_n = A$, then the product $(A \times \ldots \times A) = A \times \ldots \times A = A^n$ is uniquely determined by $A$ and the number of factors $n$.

Any subset $R$ of the product set $(A \times B)$ is called a *binary relation* on the pair $(A, B)$ or between the elements of the sets of the pair $(A, B)$. If $A = B$ we speak about a binary relation on $A$ or between the elements of $A$. If $(a, b) \in R$ we say that the relation $R$ holds for the pair $(a, b)$ and we express this fact by writing $a\,R\,b$. In practice, various other symbols might replace $a\,R\,b$, e.g., $a \perp b$, $a \parallel b$, $a \sim b$, or $a < b$, but even then it is worthwhile to interpret the relation as a particular subset $R$ of $(A \times B)$.

The *inverse* of a binary relation $R$ on an ordered pair $(A, B)$ is defined as a binary relation on the pair $(B, A)$:

$$R^{-1} = \{(b, a) : a \in A, \quad b \in B, \quad \text{and} \quad (a, b) \in R\}.$$

In the special case $A = B$ both $R$ and $R^{-1}$ are subsets of $A \times A$ and it might happen that $R = R^{-1}$ in which case $R$ is called *symmetric*. The set

$$I = \{(a, a) : a \in A\}$$

is called the *diagonal* of the product $A \times A$. If $R$ is such that $I \subseteq R$, then it is called a *reflexive* binary relation on the elements of $A$. *Antireflexivity* means that $a\,R\,a$ never holds and *antisymmetry* expresses the additional fact that at most one of the possibilities $a\,R\,b$ and $b\,R\,a$ can take place. For instance, if $A$ is the set of all straight lines in the plane, then parallelism $\|$ is a symmetric and reflexive relation while orthogonality $\perp$ is symmetric and antireflexive. Set theoretical inclusion $\subset$ gives an example of an antisymmetric binary relation on the set of all subsets $\mathscr{P}(X)$ of a set $X$. If $a\,R\,c$ whenever $a\,R\,b$ and $b\,R\,c$, then $R$ is called *transitive*. For instance, $\|$ and $\subset$ are transitive relations while $\perp$ is not.

Parallelism gives a simple example of one of the best-known types of binary relations: An *equivalence relation* is a reflexive, symmetric, and transitive relation on some set $A$. Another known type is *linear ordering*. This means an antisymmetric and transitive relation $<$ such that if $a \neq b$ then $a < b$ or $b < a$. If the last requirement is omitted we speak about an *antireflexive partial ordering*. A reflexive and transitive binary relation $\leqslant$ is called a *reflexive partial ordering*. Notice that $a \leqslant b$ and $b \leqslant a$ might hold simultaneously even if $a$ and $b$ are distinct elements of the set $A$. For instance, any equivalence relation is a reflexive partial ordering.

A *function* $f$ on a set $A$ with values in another set $B$ can be most easily defined by its graph which is a subset of the product $(A \times B)$. A relation $F \subseteq A \times B$ will be called the *graph* of a *function* $f : A \rightarrow B$ if for any $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in F$. If the sets $A$ and $B$ are distinct no confusion can arise: The function $f : A \rightarrow B$ is determined by the ordering $(A, B)$ which is now written as $(A \times B)$. It might happen that $F$ and $F^{-1}$ are both graphs in which case the associated functions are denoted by $f$ and $f^{-1}$. If $A = B$, then $f$ and $f^{-1}$ are distinct or not accordingly as $F \neq F^{-1}$ or $F = F^{-1}$. If $f^{-1}$ exists, then $f$ is called *invertible* and $f^{-1}$ is its *inverse*. By our definition $F^{-1}$ is a graph only if for every $b \in B$ there is exactly one $a \in A$ such that $(a, b) \in F$. Thus an invertible function $f : A \rightarrow B$ maps $A$ onto $B$ and as such yields a *one-to-one correspondence* between the elements of $A$ and $B$. Although a one-to-one correspondence could be viewed as a symmetric relation $A \rightleftarrows B$, it is preferable to keep the asymmetry so that a one-to-one correspondence is nothing but an invertible map $f : A \rightarrow B$. If distinct elements of $A$ are mapped into distinct elements of $B$ then $f : A \rightarrow B$ is called *injective* and if $f$ maps $A$ onto $B$ then it is called *surjective*.

## 2. Set Theoretical Equivalence and Denumerability

In intuitive set theory the existence of infinite sets is taken for granted. For instance, one can be easily convinced that the natural numbers 1, 2, 3, ... may be collected in a single family $\{1, 2, 3, \ldots\}$ which is a set. As soon as an infinite set is given, others can be constructed by elementary set theoretic operations, e.g., by taking the set of all ordered pairs or the set of all subsets of the given set. For it is assumed that the ordered pairs $(x, y)$ where $x \in X$ and $y \in Y$ can be considered as the elements of a single set $X \times Y$ and similarly there is a set $\mathscr{P}(X)$ called the *power set* of $X$ which consists of the subsets of $X$:

$$\mathscr{P}(X) = \{A : A \subseteq X\}.$$

In axiomatic set theory the existence of the sets $\{1, 2, 3, \ldots\}$, $X \times Y$, and $\mathscr{P}(X)$ for any $X$, $Y$ can be proved from the axioms. We can make an easy compromise by taking the existence of these sets as axioms.

The first significant set theoretic result of Cantor concerns a classification of infinite sets. Two sets $A$ and $B$ are called *equivalent*, or of the same *cardinality*, if there exists a one-to-one correspondence $f : A \to B$ between their elements. If such a one-to-one map exists between $A$ and $B$ we write $A \sim B$. It is clear that $A \sim A$ for any set $A$ and also that $A \sim B$ implies $B \sim A$. A simple reasoning shows that if $A \sim B$ and $B \sim C$, then $A \sim C$. Two finite sets are equivalent if and only if they have the same number of elements. The concept of equivalence is of primary importance in the case of infinite sets when it can be used to distinguish between various infinite sets.

Following Cantor we prove:

*No set $X$ is equivalent to its power set $\mathscr{P}(X)$.*

It will be sufficient to prove the following proposition: If $\mathscr{Q}$ is a subset of $\mathscr{P}(X)$ whose elements can be brought into a one-to-one correspondence $f : X \to \mathscr{Q}$ with the elements of $X$, then $\mathscr{Q}$ is a proper subset of $\mathscr{P}(X)$. In order to construct a subset $A$ of $X$ not belonging to $\mathscr{Q}$ we consider the image points $f(x)$ and distinguish between the possibilities $x \in f(x)$ and $x \notin f(x)$. Thus we define

$$A = \{x : x \in X \quad \text{and} \quad x \notin f(x)\}.$$

By the one-to-one correspondence $f : X \to \mathscr{Q}$ for every $Q$ in $\mathscr{Q}$ there is a unique $x$ in $X$ such that $f(x) = Q$. If $x \in f(x) = Q$, then $x \notin A$, so $A \neq Q$, and if $x \notin f(x) = Q$, then $x \in A$, so again $A \neq Q$. Thus $A$ is not an element of $\mathscr{Q}$ and consequently $\mathscr{Q}$ is a proper subset of $\mathscr{P}(X)$.

Cantor's theorem shows that there exist nonequivalent infinite sets. For example, $\{1, 2, 3, \ldots\}$ is not equivalent to the set of its subsets.