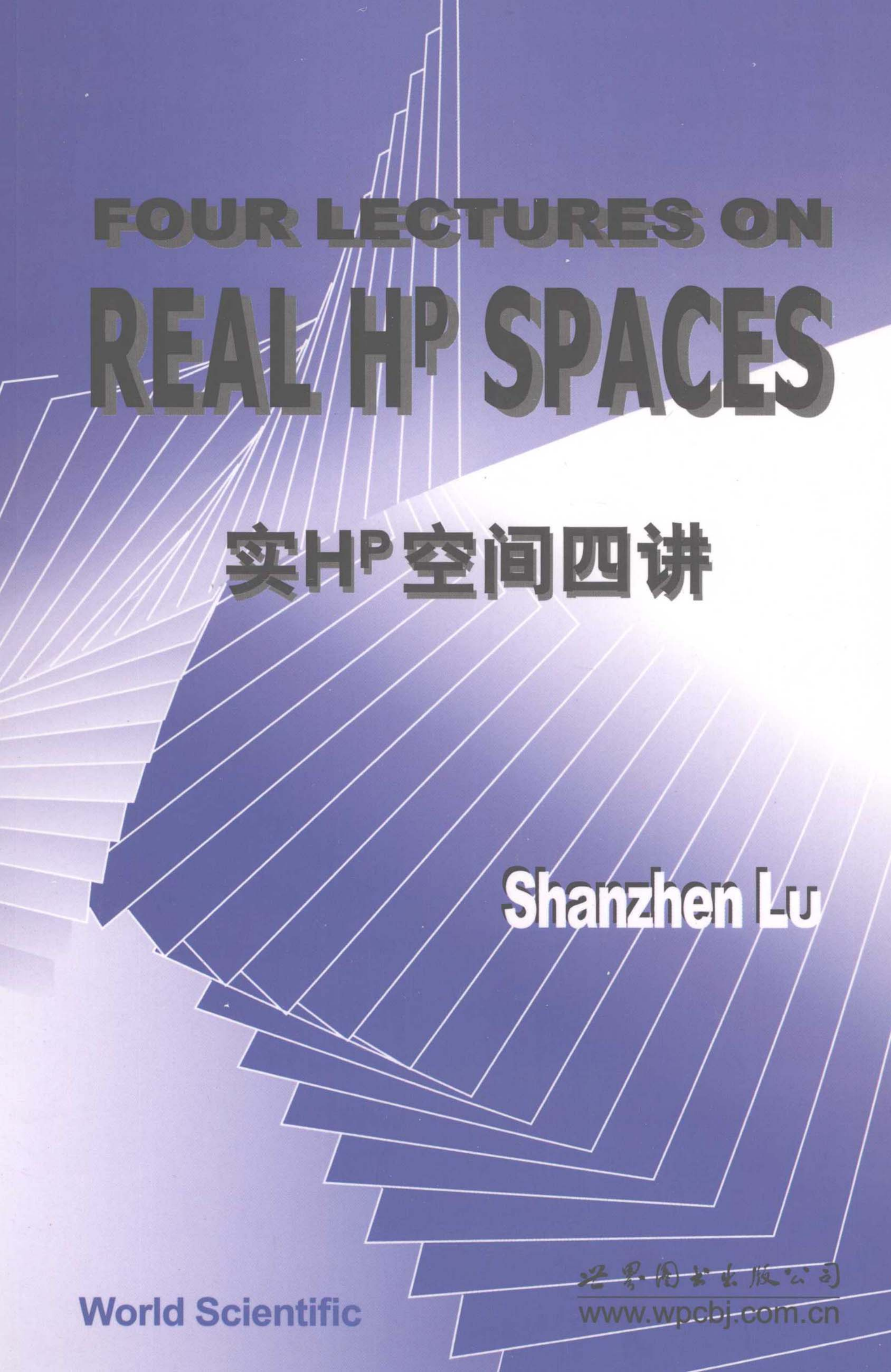

FOUR LECTURES ON
REAL H^p SPACES

实H^p空间四讲

Shanzhen Lu

World Scientific

世界图书出版公司
www.wpcbj.com.cn

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Shanzhen Lu
Department of Mathematics
Beijing Normal University

World Scientific
Singapore • New Jersey • London • Hong

Published by

World Scientific Publishing Co. Pte. Ltd.
P O Box 128, Farrer Road, Singapore 9128
USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

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ISBN 981-02-2158-4

本书由世界科技出版公司授权重印出版，限于中国大陆地区发行。

图书在版编目（C I P）数据

实Hp空间四讲=Four Lectures on Real Hp Spaces：英
文／（德）陆（Lu,S.）著. —北京：世界图书出版公司北京公
司，2008.3
ISBN 978-7-5062-9184-2

Ⅰ.实… Ⅱ.陆… Ⅲ.哈代空间－英文 Ⅳ.0177.3

中国版本图书馆CIP数据核字（2008）第021792号

书　　名：Four Lectures on Real H^p Spaces
作　　者：Shanzhen Lu
中 译 名：实H^p空间四讲
责任编辑：高蓉 刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：010-64015659
电子信箱：kjsk@vip.sina.com

开　　本：24开
印　　张：9.5
版　　次：2008 年 8月第 1 次印刷
版权登记：图字:01-2008-3052
书　　号：978-7-5062-9184-2 / O・575　　定　　价：49.00 元

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Preface

It is well known that the study on H^p spaces has been going on for a long period. The classical H^p spaces on the unit circle or upper half-plane are defined by the aid of complex method. The theory of these spaces plays an important role in the study of the classical Fourier analysis. It is natural to extend the definitions of these spaces to higher dimensional case along with the development of the Fourier analysis on Euclidean spaces. The first work on this was done by E. M. Stein and G. Weiss. The definition and theory of the n-dimensional H^p spaces that they established in the early days of the sixties are based on the method of harmonic functions instead of the complex method. However, the most important step in the development of H^p spaces is that the real variable theory of H^p spaces was found by virtue of the method of maximal functions in the early days of the seventies. The purpose of this book is to introduce the real variable theory of H^p spaces in short and pay more attention to its applications to some respects in analysis fields.

The whole book consists of four chapters. The basic theory of Fefferman-Stein on real H^p spaces is briefly introduced in Chapter 1. The contents in Chapter 2 involve the atomic decomposition theory and the molecular decomposition theory of real H^p spaces. In addition, the dual spaces of real H^p spaces, the interpolation of operators in H^p spaces, and the interpolation of H^p spaces are also discussed in Chapter 2 as a prerequisite for Chapters 3 and 4. The properties of several basic operators in H^p spaces will be discussed in Chapter 3 in detail. Among them, some basic results are contributed by Chinese mathematicians, such as the decomposition theory of weak H^p spaces and its applications to the study on the sharpness of singular integrals, a new method to deal with the elliptic Riesz means in H^p spaces, and the transference theorem of H^p multipliers, etc. The last chapter is devoted to applications of real H^p spaces to approximation theory. The materials in Chapter 4 are fully contributed by Chinese mathematicians.

I wish to express my thanks to Professor Guido Weiss who introduced me to this field in the early days of the eighties. I am also grateful to Professor M. T. Cheng and Professor Y. S. Sun for their constant encouragement. In addition, I wish to express my gratitude to all my students during the period of graduate courses I gave at Beijing Normal University from 1983 to 1992: Dr. K. Y. Wang, Dr. Z. X. Liu, Dr. H. P. Liu, Dr. Y. Zhang, Mr. Y. S. Jiang, Mr. L. X. Dai, and Mr. G. L. Chen, who made many valuable

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suggestions; Dr. D. C. Yang, who prepared a lecture notes in English on which the manuscript of this book is based; also Dr. Y. Ding and Dr. B. L. Ma, who completed the typewriting of the manuscript.

Shanzhen Lu
Beijing Normal University

Contents

Chapter 1

Real Variable Theory of $H^p(\mathbb{R}^n)$ Spaces

§1. Definition of $H^p(\mathbb{R}^n)$ spaces

The classical H^p space is defined for some analytic functions (cf. Duren [Du 1] or Zygmund [Zy 1]). The so-called space $H^p(\mathbb{R}^2_+)$ consists of those functions F which are analytic in the upper half plane $\mathbb{R}^2_+$ and satisfy

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dx < \infty, 0 < p < \infty.$$

It is not difficult to prove that if $F \in H^p(\mathbb{R}^2_+)$, the boundary value of the real part of F,

$$\lim_{y \to 0} \mathrm{Re}\{F(x+iy)\},$$

exists *a.e.* (In general, such boundary value is a distribution on $\mathbb{R}$). Thus, we can define a space by gathering all boundary values of functions in $H^p(\mathbb{R}^2_+)$, which is called the real $H^p(\mathbb{R})$ space. That is

$$\mathrm{Re}H^p(\mathbb{R}) = \{f : f(x) = \lim_{y \to 0} \mathrm{Re}\{F(x+iy)\}, F \in H^p(\mathbb{R}^2_+)\}.$$

It will be found later that $\mathrm{Re}H^p(\mathbb{R})$ is identical with $L^p(\mathbb{R})$ for $p > 1$ and is totally different from $L^p(\mathbb{R})$ for $0 < p \leq 1$.

Along with the development of n dimensional Fourier analysis, there exists naturally a problem on extending $\mathrm{Re}H^p(\mathbb{R})$ to the version of n dimension. E. M. Stein and G. Weiss proposed the concept of the generalized Cauchy-Riemann equations, basing on the fact that the real and imaginary part of function in $H^p(\mathbb{R}^2_+)$ satisfy the Cauchy-Riemann condition, and defined $H^p(\mathbb{R}^{n+1}_+)$ spaces (cf. E. M. Stein and G. Weiss [SW1]). Precisely, let us consider a system of harmonic functions on $\mathbb{R}^n \times (0, \infty)$

$$F(x_1, \cdots, x_n, y) = (u_0(x_1, \cdots, x_n, y), \cdots, u_n(x_1, \cdots, x_n, y))$$

satisfying the generalized Cauchy–Riemann equations

$$\begin{cases} \dfrac{\partial u_j}{\partial x_i} = \dfrac{\partial u_i}{\partial x_j}, \quad 0 \le i, j \le n, \\ \displaystyle\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \end{cases} \tag{1.1}$$

where $x_0 = y$. Now, let us define

$$H^p(\mathbb{R}^{n+1}_+) = \left\{ F : F \text{ satisfies (1.1) and } \sup_{0<y<\infty} \int_{\mathbb{R}^n} |F(x,y)|^p \, dx < \infty \right\}.$$

Similarly, we then use the boundary values of the first component of elements in $H^p(\mathbb{R}^{n+1}_+)$ to define the real $H^p(\mathbb{R}^n)$ space as follows

$$\mathrm{Re}H^p(\mathbb{R}^n) = \{ f : f(x) = \lim_{y \to 0} u_0(x,y), F \in H^p(\mathbb{R}^{n+1}_+) \}.$$

It should be pointed out that Stein and Weiss define $\mathrm{Re}H^p(\mathbb{R}^n)$ only for $p > (n-1)/n$. Afterwards, A. P. Calderón and A. Zygmund [CZ 1] removed this limit of p and this yielded a definition of $\mathrm{Re}H^p(\mathbb{R}^n)$ spaces for all $p, 0 < p < \infty$, which are now called the real Hardy spaces.

So far, we notice that $\mathrm{Re}H^p(\mathbb{R})$ is defined for a class of analytic functions, while the definition of $\mathrm{Re}H^p(\mathbb{R}^n)$ is also evolved from the properties of analytic functions. In a sense, the definition of $\mathrm{Re}H^p(\mathbb{R}^n)$ is closely related with analytic functions. In the 1970's, an important fact concerning the real variable characters of $\mathrm{Re}H^p$ spaces was discovered. In fact, Hardy and Littlewood pointed out long ago that if $f \in \mathrm{Re}H^p(\mathbb{R})$, then its Poisson nontangential maximal function $P^*_\nabla(f)(x) := \sup_{|y-x|<t} |(f * P_t)(y)| \in L^p(\mathbb{R})$. In 1971, D. L. Burkholder, R. F. Gundy and M. L. Silverstein [BGS 1] proved its converse proposition. Thus, $f \in \mathrm{Re}H^p(\mathbb{R})$ if and only if the Poisson nontangential maximal function of $f, P^*_\nabla(f)$, belongs to $L^p(\mathbb{R})$. Obviously, this character of $\mathrm{Re}H^p(\mathbb{R})$ is completely independent of the definition described above that uses the properties of analytic functions. It is interesting to observe that the methoud used in [BGS 1] is of probabilistic. In 1972, C. Fefferman and E. M. Stein [FS 1] extended the above character to the version of n dimension by real variable methods and obtained an equivalent definition of $\mathrm{Re}H^p(\mathbb{R}^n)$ spaces.

Definition 1.1. Let f be a tempered distribution on $\mathbb{R}^n$ and P the Poisson kernel. If the Poisson nontangential maximal function $P^*_\nabla(f)(x) := \sup_{|y-x|<t} |(f * P_t)(y)| \in L^p(\mathbb{R}^n)$, then we say $f \in \mathrm{Re}H^p(\mathbb{R}^n)$, where the set $\{(y,t) : |y-x| < t\}$ is a cone in $\mathbb{R}^{n+1}_+ = \{(y,t) : y \in \mathbb{R}^n, t > 0\}$, and

$P_t(x) = t^{-n}P(x/t)$. For simplicity, $\mathrm{Re}H^p(\mathbb{R}^n)$ will be written as $H^p(\mathbb{R}^n)$ in the following.

Remark 1.1. Observe that $P \notin \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz function class. Thus, for a tempered distribution f, its Poisson convolution may be senseless. In fact, f in the definition 1.1 should belong to a distribution class satisfying certain increasing condition. In order to explain this point, it is necessary to introduce a bigger test function space than $\mathcal{S}(\mathbb{R}^n)$ such that the Poisson kernel belongs to it, and its dual space is exactly the distribution class mentioned above. Let us now write

$$D_{L^1} = \{\varphi \in C^\infty(\mathbb{R}^n) : D^\alpha\varphi \in L^1(\mathbb{R}^n), \forall\alpha\},$$

where α is an n tuple index and D^α is a differential operator of degree α. The dual spaces of $D_{L^1}, (D_{L^1})'$, has the following relation with $\mathcal{S}'$ (cf. J. Baros-Neto [B 1]): $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if there exists a nonnegative integer $k = k(f)$ such that $(1 + |x|^2)^{-k/2} \in (D_{L^1})'$. Consequently, elements in $(D_{L^1})'$ have certain increasing condition compared with one in $\mathcal{S}'$. While $\mathcal{S}'(\mathbb{R}^n)$ is called the distribution class, $(D_{L^1})'$ is a distributional class with certain increasing condition. Moreover, it is easy to see that $P \in D_{L^1}$. Hence, for any $f \in (D_{L^1})', f * P_t$ is well defined. In other words, f in Definition 1.1 should be a distribution satisfying certain increasing condition.

Having Definition 1.1, a natural extension of $H^p(\mathbb{R}^n)$ space is the following space

$$H(p,q,\mathbb{R}^n) = \{f \in \mathcal{S}' : P_\nabla^*(f) \in L(p,q,\mathbb{R}^n)\}$$

defined by C. Fefferman, N. M. Riviere, and Y. Sagher in [FRS 1], where $L(p,q,\mathbb{R}^n)$ is the Lorentz space on $\mathbb{R}^n$ (see [SW 2] for the definition). Obviously, $H(p,p,\mathbb{R}^n) = H^p(\mathbb{R}^n)$, and $H^p(\mathbb{R}^n) \subset H(p,q,\mathbb{R}^n)$ if $q > p$. $H(p,q,\mathbb{R}^n)$ is usually called the Lorentz-Hardy space.

§2. Non-tangential maximal functions

By Definition 1.1, it is easy to see that this definition does not completely get rid of the dependence on harmonic functions because of the Poisson kernel appearing in the definition. Therefore, the following problem is then posed: can the Poisson kernel in the definition 1.1 be replaced by any other kernel of approximations to the identity ? C. Fefferman and E. M. Stein gave out an affirmative answer in [FS 1]. They introduced the nontangential maximal function associated with smooth kernel as follows.

Definition 2.1. Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int \varphi(x)\,dx = 1$. If we write

$$\varphi_\nabla^*(f)(x) = \sup_{|y-x|<t} |(f * \varphi_t)(y)|,$$

then $\varphi_\nabla^*(f)$ is called the φ-nontangential maximal function of f.

In order to reveal the relation between $\varphi_\nabla^*(f)$ and $P_\nabla^*(f)$, we need to set up several lemmas.

Lemma 2.1. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int \varphi(x)\,dx = 1$. For any $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $N \in \mathbf{N}$, there exist $\theta^{(t)} \in \mathcal{S}(\mathbb{R}^n)(0 < t < 1)$ and $m \in \mathbf{N}$ such that

(i) $\psi(x) = \int_0^1 (\varphi_t * \theta^{(t)})(x)\,dt$,

(ii) $\int_{\mathbf{R}^n} (1+|x|)^N |\theta^{(t)}(x)|\,dx \le C t^N \sup_{u\in\mathbf{R}^n, |\alpha|\le m} (1+|u|)^{m+n} |D^\alpha \psi(u)|$,

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and

$$D^\alpha \psi(u) = \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}} \psi(u_1, \cdots, u_n).$$

Proof. Choose a $\zeta \in C^\infty(0,1)$ such that

$$\begin{aligned} &\zeta(t) = t^N/N!, && \text{if } 0 \le t \le 1/2, \\ &0 \le \zeta(t) \le t^N/N!, && \text{if } 1/2 < t \le 1, \end{aligned}$$

and $\zeta^{(j)}(1) = 0, 0 \le j \le N+1$. Note that

$$\begin{aligned} \frac{\partial^{N+1}}{\partial t^{N+1}} (\underbrace{\varphi_t * \cdots * \varphi_t}_{N+2\,\text{times}}) = &\sum_{i_1+\cdots+i_{N+1}=N+1} C_{i_1,\cdots,i_{N+1}} \\ &\times \varphi_t * \frac{\partial^{i_1}\varphi_t}{\partial t^{i_1}} * \cdots * \frac{\partial^{i_{N+1}}\varphi_t}{\partial t^{i_{N+1}}}, \end{aligned}$$

where $i_j (1 \le j \le N+1)$ are nonnegative integers. Hence, if we set

$$\begin{aligned} \theta^{(t)}(x) = &(-1)^{N+1}\zeta(t) \sum_{i_1+\cdots+i_{N+1}=N+1} C_{i_1,\cdots,i_{N+1}} \\ &\times \left(\frac{\partial^{i_1}\varphi_t}{\partial t^{i_1}} * \cdots * \frac{\partial^{i_{N+1}}\varphi_t}{\partial t^{i_{N+1}}} * \psi \right)(x) - \zeta^{(N+1)}(t) (\underbrace{\varphi_t * \cdots * \varphi_t}_{N+1\,\text{times}} * \psi)(x), \end{aligned}$$

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then it is easy to verify that $\theta^{(t)}$ satisfies (i) and (ii). In fact, using integration by parts $N+1$ times to

$$I = (-1)^{N+1} \int_0^1 \zeta(t) \frac{\partial^{N+1}}{\partial t^{N+1}} (\underbrace{\varphi_t * \cdots * \varphi_t}_{N+2 \text{ times}}) * \psi \, dt,$$

we obtain

$$I = \underbrace{\varphi_t * \cdots * \varphi_t}_{N+2 \text{ times}} * \psi \Big|_{t=0} + \int_0^1 \zeta^{(N+1)}(t) \underbrace{\varphi_t * \cdots * \varphi_t}_{N+2 \text{ times}} * \psi \, dt.$$

Observing that

$$\varphi_t * \cdots * \varphi_t * \psi \Big|_{t=0} = \lim_{t \to 0_+} \psi * (\varphi * \cdots * \varphi)_t = \psi,$$

we have

$$\psi = I - \int_0^1 \zeta^{(N+1)}(t) \underbrace{\varphi_t * \cdots * \varphi_t}_{N+2 \text{ times}} * \psi \, dt = \int_0^1 \varphi_t * \theta^{(t)} \, dt.$$

Thus, (i) is satisfied. To prove (ii), it is suffices to show that there exists $m \in \mathbf{N}$ such that

$$\begin{aligned} &\sup_{0<t<1} \int_{\mathbf{R}^n} (1+|x|)^N \left| \left(\frac{\partial^{i_1} \varphi_t}{\partial t^{i_1}} * \cdots * \frac{\partial^{i_{N+1}} \varphi_t}{\partial t^{i_{N+1}}} * \psi \right)(x) \right| dx \\ &\leq C \sup_{u \in \mathbf{R}^n, |\alpha| \leq m} (1+|u|)^{m+n} |D^\alpha \psi(u)|, \end{aligned}$$

and

$$\begin{aligned} &\sup_{0<t<1} \int_{\mathbf{R}^n} (1+|x|)^N |(\underbrace{\varphi_t * \cdots * \varphi_t}_{N+1 \text{ times}} * \psi)(x)| \, dx \\ &\leq C \sup_{u \in \mathbf{R}^n, |\alpha| \leq m} (1+|u|)^{m+n} |D^\alpha \psi(u)|. \end{aligned}$$

We only prove the first one in the following. Choose $m \geq N+1$, then

$$\begin{aligned}
&\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}\right)(x)\right|dx\\
&=\int_{\mathbf{R}^n}(1+|x|)^N\left|\frac{\partial^{i_1}}{\partial t_1^{i_1}}(\psi*\varphi_{t_1})(x)\right|dx\\
&=\int_{\mathbf{R}^n}(1+|x|)^N\left|\int_{\mathbf{R}^n}\frac{\partial^{i_1}}{\partial t_1^{i_1}}\psi(x-t_1y)\varphi(y)\,dy\right|dx\\
&\leq \sup_{u\in\mathbf{R}^n,|\alpha|\leq i_1}(1+|u|)^{m+n}|D^\alpha\psi(u)|\\
&\quad\times\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{(1+|x|)^N(1+|y|)^{i_1}}{(1+|x-t_1y|)^{m+n}}|\varphi(y)|\,dxdy\\
&\leq \sup_{u\in\mathbf{R}^n,|\alpha|\leq i_1}(1+|u|)^{m+n}|D^\alpha\psi(u)|\\
&\quad\times\int_{\mathbf{R}^n}(1+|y|)^{N+i_1}|\varphi(y)|\left(\int_{\mathbf{R}^n}\frac{dx}{(1+|x-t_1y|)^{m+n-N}}\right)dy.
\end{aligned}$$

From this, it follows

$$\begin{aligned}
\sup_{0<t_1<1}\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}\right)(x)\right|dx\\
\leq C\sup_{u\in\mathbf{R}^n,|\alpha|\leq i_1}(1+|u|)^{m+n}|D^\alpha\psi(u)|.
\end{aligned}$$

Let us now consider

$$\sup_{0<t_1,t_2<1}\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}*\frac{\partial^{i_2}\varphi_{t_2}}{\partial t_2^{i_2}}\right)(x)\right|dx.$$

If we set $\psi_1(x)=\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}\right)(x)$ and use the inequality proved above, then

$$\begin{aligned}
\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}*\frac{\partial^{i_2}\varphi_{t_2}}{\partial t_2^{i_2}}\right)(x)\right|dx\\
\leq C\sup_{u\in\mathbf{R}^n,|\alpha|\leq i_2}(1+|u|)^{m+n}|D^\alpha\psi_1(u)|.
\end{aligned}$$

Observe that $D^\alpha\psi_1(u)=\left(D^\alpha\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}\right)(u)$. Thus, if $|\alpha|\leq i_2$, we have

$$\begin{aligned}
&(1+|u|)^{m+n}|D^\alpha\psi_1(u)| \\
&= (1+|u|)^{m+n}\left|\frac{\partial^{i_1}}{\partial t_1^{i_1}}(D^\alpha\psi * \varphi_{t_1})(u)\right| \\
&= (1+|u|)^{m+n}\left|\int_{\mathbf{R}^n}\frac{\partial^{i_1}}{\partial t_1^{i_1}}(D^\alpha\psi)(u-t_1y)\varphi(y)\,dy\right| \\
&\le \sup_{w\in\mathbf{R}^n,|\beta|\le i_1+i_2}(1+|w|)^{m+n}|D^\beta\psi(w)| \\
&\times\int_{\mathbf{R}^n}\frac{(1+|u|)^{m+n}(1+|y|)^{i_1}|\varphi(y)|}{(1+|u-t_1y|)^{m+n}}\,dy \\
&\le \sup_{w\in\mathbf{R}^n,|\beta|\le i_1+i_2}(1+|w|)^{m+n}|D^\beta\psi(w)|\int_{\mathbf{R}^n}(1+|y|)^{m+n+i_1}|\varphi(y)|\,dy.
\end{aligned}$$

From the above two inequalities, it follows

$$\begin{aligned}
&\sup_{0<t_1,t_2<1}\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}*\frac{\partial^{i_2}\varphi_{t_2}}{\partial t_2^{i_2}}\right)(x)\right|dx \\
&\le C\sup_{u\in\mathbf{R}^n,|\alpha|\le i_1+i_2}(1+|u|)^{m+n}|D^\alpha\psi(u)|.
\end{aligned}$$

In general case, by a similar method, we can obtain

$$\begin{aligned}
&\sup_{0<t_1,\cdots,t_{N+1}<1}\int_{\mathbf{R}^n}(1+|x|)^N\left|\left(\psi*\frac{\partial^{i_1}\varphi_{t_1}}{\partial t_1^{i_1}}*\cdots*\frac{\partial^{i_{N+1}}\varphi_{t_{N+1}}}{\partial t_{N+1}^{i_{N+1}}}\right)(x)\right|dx \\
&\le C\sup_{u\in\mathbf{R}^n,|\alpha|\le i_1+\cdots+i_{N+1}=N+1}(1+|u|)^{m+n}|D^\alpha\psi(u)|.
\end{aligned}$$

Taking $t_1 = \cdots = t_{N+1} = t$ in this inequality, we then obtain the desired inequality.

Lemma 2.2. If $P_\nabla^*(f)\in L^p(\mathbb{R}^n), 0<p<\infty$, then there exists a $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\int\varphi(x)\,dx = 1$ and the radial maximal function

$$\varphi_+^*(f)(x) := \sup_{t>0}|(f*\varphi_t)(x)|\in L^p(\mathbb{R}^n).$$

Proof. Choose a $\psi\in C^\infty[1,\infty)\cap L^1[1,\infty)$ such that

$$\int_1^\infty s^k\psi(s)\,ds = \begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } k\in\mathbf{N}.\end{cases}$$

It should be pointed out that the above ψ exists (cf. E. M. Stein [St 2, p.182]). We now let

$$\varphi(x) = \int_1^\infty \psi(s)P_s(x)\,ds.$$

To verify $\varphi \in \mathcal{S}$, we need only to prove $\widehat{\varphi} \in \mathcal{S}$. In fact,

$$\widehat{\varphi}(\xi) = \int \varphi(x)e^{-2\pi i x\cdot\xi}\,dx = \int_1^\infty \psi(s)\widehat{P}_s(\xi)\,ds = \int_1^\infty \psi(s)e^{-s|\xi|}\,ds.$$

Hence, by properties of the exponential function, it is clear that $\widehat{\varphi}(\xi)$ is rapidly decreasing at the infinity and is smooth away from the origin. Moreover, by the following asymptotic expansion

$$\widehat{\varphi}(\xi) = \sum_{k=0}^{N-1}(-1)^k\frac{|\xi|^k}{k!}\int_1^\infty s^k\psi(s)\,ds + O(|\xi|^N)$$

and the arbitrariness of $N, \widehat{\varphi}$ is also smooth at the origin. Thus, $\widehat{\varphi} \in \mathcal{S}$. Next, it is easy to see that

$$\int \varphi(x)\,dx = \widehat{\varphi}(0) = 1$$

and

$$\begin{aligned}\varphi_+^*(f)(x) &= \sup_{t>0}|(f*\varphi_t)(x)|\\ &= \sup_{t>0}\left|\int_1^\infty \psi(s)(f*P_{ts})(x)\,ds\right|\\ &\leq \int_1^\infty |\psi(s)|\,dsP_\nabla^*(f)(x).\end{aligned}$$

This finishes the proof of Lemma 2.2.

Lemma 2.3. Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n), \int\varphi(x)\,dx = 1$, and $0 < p < \infty$. If $\varphi_+^*(f) \in L^p(\mathbb{R}^n)$, then

$$\|\varphi_\nabla^*(f)\|_p \leq C_{p,n}\|\varphi_+^*(f)\|_p.$$

Proof. By Fatou's lemma, we need only to prove

$$\int_{\mathbf{R}^n}\left\{\sup_{|x-y|<t<1/\varepsilon}\left|(f*\varphi_t)(y)\left(\frac{t}{t+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\right|\right\}^p dx$$
$$\le C_{p,n}\|\varphi_+^*(f)\|_p^p. \tag{2.1}$$

To do this, let us now set

$$u_{\varepsilon,N}^* = \sup_{|x-y|<t<1/\varepsilon}|(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N},$$

where $0<\varepsilon<1$. We first point out in the following that $u_{\varepsilon,N}^*\in L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, if N is big enough. Consider $(f*\varphi_t)(y)$ as a bounded linear functional on $\mathcal{S}$. Thus, there exist nonnegative integers l and m such that

$$|(f*\varphi_t)(y)|\le C\sup_{x\in\mathbf{R}^n,|\beta|\le m}(1+|x|)^l|D_x^\beta\varphi_t(y-x)|,$$

where D_x^β is a differential operator of degree β, and C is independent of t,y and φ (see E. M. Stein, G. Weiss [SW 2]). From the elementary inequality

$$\left(\frac{1+|x|}{1+|y|}\right)^r\le(1+|x-y|)^r,\qquad r\ge 0,$$

it follows

$$|(f*\varphi_t)(y)|\le C(1+|y|)^l\sup_{u\in\mathbf{R}^n,|\beta|\le m}(1+|u|)^l|D_u^\beta\varphi_t(u)|.$$

By the estimate

$$\sup_{u\in\mathbf{R}^n}(1+|u|)^l|D_u^\beta\varphi_t(u)|\le\begin{cases}t^{-n-|\beta|}\|\varphi\|, & \text{if } t<1,\\ t^{-n-|\beta|+l}\|\varphi\|, & \text{if } 1\le t<1/\varepsilon,\end{cases}$$

where $|\beta|\le m$ and $\|\varphi\|=\sup_{v\in\mathbf{R}^n,|\beta|\le m}(1+|v|)^l|D^\beta\varphi(v)|$, we have

$$|(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\le C_\varepsilon(1+|y|)^{-N+l},$$

if N is big enough. Hence, $u_{\varepsilon,N}^*\in L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Let us now denote

$$V_{\varepsilon,N,M}^*(x)=\sup_{y\in\mathbf{R}^n,t<1/\varepsilon}|(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N(1+\varepsilon|y|)^{-N}\left(\frac{t}{|x-y|+t}\right)^M,$$

and

$$U^*_{\varepsilon,N}(x) = \sup_{|x-y|<t<1/\varepsilon} t|\nabla_y(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N (1+\varepsilon|y|)^{-N}.$$

It is not hard to show that

$$\|V^*_{\varepsilon,N,M}\|_p \le C_{p,n}\|u^*_{\varepsilon,N}\|_p, \qquad \text{if } M > n/p, \tag{2.2}$$

and

$$U^*_{\varepsilon,N}(x) \le CV^*_{\varepsilon,N,M}(x). \tag{2.3}$$

In fact, from the definition of $u^*_{\varepsilon,N}$, it follows

$$|(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N (1+\varepsilon|y|)^{-N} \le u^*_{\varepsilon,N}(z), \qquad \text{if } z \in B(y,t).$$

Since $B(y,t) \subset B(x, |x-y|+t)$, for any $p_1 (0 < p_1 < \infty)$ we have

$$\begin{aligned}
&\left\{|(f*\varphi_t)(y)|\left(\frac{t}{t+\varepsilon}\right)^N (1+\varepsilon|y|)^{-N}\right\}^{p_1} \\
&\qquad \le \frac{1}{|B(y,t)|}\int_{B(y,t)} \{u^*_{\varepsilon,N}(z)\}^{p_1}\,dz \\
&\qquad \le C\left(\frac{|x-y|+t}{t}\right)^n \frac{1}{|B(x,|x-y|+t)|}\int_{B(x,|x-y|+t)} \{u^*_{\varepsilon,N}(z)\}^{p_1}\,dz.
\end{aligned}$$

Therefore, we have

$$\{V^*_{\varepsilon,N,M}(x)\}^{p_1} \le CHL({u^*_{\varepsilon,N}}^{p_1})(x), \qquad \text{if } M > \frac{n}{p_1},$$

where $HL(f)(x)$ is the Hardy-Littlewood maximal function of f. Consequently, (2.2) is deduced by this inequality and the properties of the HL maximal function. Let us turn to prove (2.3). Since

$$t|\nabla_y(f*\varphi_t)(y)| = |f*(\nabla\varphi)_t(y)|$$

and $\nabla\varphi \in \mathcal{S}$, it follows from Lemma 2.1 that

$$\nabla\varphi = \int_0^1 \varphi_\tau * \theta^{(\tau)}\,d\tau,$$

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