CORNELIUS LANCZOS

The Variational Principles of Mechanics

MATHEMATICAL EXPOSITIONS, No. 4

THE VARIATIONAL PRINCIPLES OF MECHANICS

by

CORNELIUS LANCZOS

UNIVERSITY OF TORONTO PRESS
TORONTO

PREFACE

FOR a number of years the author has conducted a two-semester lecture course on the variational principles of mechanics at the Graduate School of Purdue University. Again and again he experienced the extraordinary elation of mind which accompanies a preoccupation with the basic principles and methods of analytical mechanics. There is hardly any other branch of the mathematical sciences in which abstract mathematical speculation and concrete physical evidence go so beautifully together and complement each other so perfectly. It is no accident that the principles of mechanics had the greatest fascination for many of the outstanding figures of mathematics and physics. Nor is it an accident that the European universities of earlier days included a course in theoretical mechanics in the study plan of every prospective mathematician and physicist. Analytical mechanics is much more than an efficient tool for the solution of dynamical problems that we encounter in physics and engineering. However great may be the importance of the gyroscope as a practical instrument of navigation or engineering, it is not needed as an excuse to demonstrate the importance of theoretical mechanics. The very existence of the general principles of mechanics is their justification.

The present treatise on the variational principles of mechanics should not be regarded as competing with the standard textbooks on advanced mechanics. Without questioning the excellent quality of these primarily technical and formalistic treatments, the author feels that there is room for monographs which exhibit the fundamental skeletons of the exact sciences in an elementary and philosophically oriented fashion.

Many of the scientific treatises of today are formulated in a half-mystical language, as though to impress the reader with the uncomfortable feeling that he is in the permanent presence of a superman. The present book is conceived in a humble spirit and is written for humble people. The author knows from past experience that one outstanding weakness of our present system of college education is the custom of classing certain fundamental and apparently simple concepts as "elementary," and of relegating them to an age-level at which the student's mind is not mature enough to grasp their true meaning. The fruits of this error can be observed daily. The student who is thoroughly acquainted with the smallest details of an atom-smashing apparatus often has entirely confused ideas concerning the difference between mass and weight, or between heavy mass and inertial mass. In mechanics, which is a fundamental science, the confusion is particularly conspicuous.

To a philosophically trained mind, the difference between actual and virtual displacement appears entirely obvious and needs no further comment. But a student of today is anything but philosophically minded. To him the difference is not only not obvious, but he cannot grasp the meaning of a "virtual displacement" without experimenting with the concept for a long time by applying it to a variety of familiar situations. Hence, the author could think of no better introduction to the application of the calculus of variations than letting the student deduce a number of familiar results of vectorial mechanics from the principle of virtual work. As a by-product of these exercises, the student notices how previously unconnected and more or less axiomatically stated properties of forces and moments all follow from one single, all-embracing principle. His interest is now aroused and he would like to go beyond the realm of statics. Here d'Alembert's principle comes to his aid, showing him how the same method of virtual displacements can serve to obtain the equations of motion of an arbitrarily complicated dynamical problem.

The author is well aware that he could have shortened his exposition considerably, had he started directly with the Lagrangian equations of motion and then proceeded to Hamilton's theory. This procedure would have been justified had the purpose of this book been primarily to familiarize the student with a certain formalism and technique in writing down the differential equations which govern a given dynamical problem, together with certain "recipes" which he might apply in order to solve them. But this is exactly what the author did *not* want to do. There is a tremendous treasure of philosophical meaning behind the great theories of Euler and Lagrange, and of Hamilton and Jacobi, which is completely smothered in a purely formalistic treatment, although it cannot fail to be a source of the greatest intellectual enjoyment to every mathematically-minded person. To give the student a chance to discover for himself the hidden beauty of these theories was one of the foremost intentions of the author. For this purpose he had to lead the reader through the entire historical development, starting from the very beginning, and felt compelled to include problems which familiarize the student with the new concepts. These problems, of a simple character, were chosen in order to exhibit the general principles involved.

As regards content, an important topic not included in the present book is the perturbation theory of the dynamical equations. Also, an originally planned chapter on Relativistic Mechanics had to be omitted, owing to limitations of space. Yet even the material as it stands can well form the subject matter of a two-semester graduate course of three hours weekly, and it may suffice for a student of mathematics or physics or engineering who does not intend to specialize in mechanics, but wants to get a thorough grasp of the underlying principles.

The author apologizes for not giving many specific references. This material has grown with him for many years and consequently he is often unable to tell where he got his information. He is primarily indebted to Whittaker's *Analytical Dynamics*, Nordheim's article in the *Handbuch der Physik*, and Prange's article in the *Encyklopaedie der mathematischen Wissenschaften*—all of which are mentioned in the Bibliography and recommended for advanced reading.

The author is deeply indebted to Professor J. L. Synge for the painstaking care with which he reviewed the manuscript and pointed out many weak points in the author's presentation. In

some instances a divergence of viewpoints was apparent, but in others an improved formulation could be given, thus enhancing substantially the readability of the book. The author is also indebted to Professor G. de B. Robinson and Professor A. F. C. Stevenson who revised the entire manuscript, Mrs. Ida Rhodes of the Mathematical Tables Project, New York City, and Mr. G. F. D. Duff of the University of Toronto, who helped read the proof and prepare the Index. The generous co-operation of all these friends and colleagues has been a heart-warming experience.

The variational principles of mechanics are firmly rooted in the soil of that great century of Liberalism which starts with Descartes and ends with the French Revolution and which has witnessed the lives of Leibniz, Spinoza, Goethe, and Johann Sebastian Bach. It is the only period of cosmic thinking in the entire history of Europe since the time of the Greeks. If the author has succeeded in conveying an inkling of that cosmic spirit, his efforts will be amply rewarded.

CORNELIUS LANCZOS

Los Angeles
May, 1949

INTRODUCTION

1. The variational approach to mechanics. Ever since Newton laid the solid foundation of dynamics by formulating the laws of motion, the science of mechanics developed along two main lines. One branch, which we shall call "vectorial mechanics,"[1] starts directly from Newton's laws of motion. It aims at recognizing all the forces acting on any given particle, its motion being uniquely determined by the known forces acting on it at every instant. The analysis and synthesis of forces and moments is thus the basic concern of vectorial mechanics.

While in Newton's mechanics the action of a force is measured by the momentum produced by that force, the great philosopher and universalist Leibniz, a contemporary of Newton, advocated another quantity, the *vis viva* (living force), as the proper gauge for the dynamical action of a force. This *vis viva* of Leibniz coincides—apart from the unessential factor 2—with the quantity we call today "kinetic energy." Thus Leibniz replaced the "momentum" of Newton by the "kinetic energy." At the same time he replaced the "force" of Newton by the "work of the force." This "work of the force" was later replaced by a still more basic quantity, the "work function." Leibniz is thus the originator of that second branch of mechanics, usually called "analytical mechanics,"[2] which bases the entire study of equilibrium and motion on two fundamental scalar quantities, the "kinetic energy" and the "work function," the latter frequently replaceable by the "potential energy."

Since motion is by its very nature a *directed* phenomenon, it seems puzzling that two scalar quantities should be sufficient to determine the motion. The energy theorem, which states that the sum of the kinetic and potential energies remains unchanged

[1]This use of the term does not necessarily imply that vectorial methods are used.

[2]See note on terminology at end of chap. I, section 1.

during the motion, yields only *one* equation, while the motion of a single particle in space requires *three* equations; in the case of mechanical systems composed of two or more particles the discrepancy becomes even greater. And yet it is a fact that these two fundamental scalars contain the complete dynamics of even the most complicated material system, provided they are used as the basis of a *principle* rather than of an *equation.*

2. The procedure of Euler and Lagrange. In order to see how this occurs, let us think of a particle which is at a point P_1 at a time t_1. Let us assume that we know its velocity at that time. Let us further assume that we know that the particle will be at a point P_2 after a given time has elapsed. Although we do not know the path taken by the particle, it is possible to establish that path completely by mathematical experimentation, provided that the kinetic and potential energies of the particle are given for any possible velocity and any possible position.

Euler and Lagrange, the first discoverers of the exact principle of least action, proceed as follows. Let us connect the two points P_1 and P_2 by *any* tentative path. In all probability this path, which can be chosen as an arbitrary continuous curve, will *not* coincide with the actual path that nature has chosen for the motion. However, we can gradually *correct* our tentative solution and eventually arrive at a curve which can be designated as the *actual* path of motion.

For this purpose we let the particle move along the tentative path in accordance with the energy principle. The sum of the kinetic and potential energies is kept constant and always equal to that value E which the actual motion has revealed at time t_1. This restriction assigns a definite velocity to any point of our path and thus determines the motion. We can choose our path freely, but once this is done the conservation of energy determines the motion uniquely.

In particular, we can calculate the time at which the particle will pass an arbitrarily given point of our fictitious path and hence the time-integral of the *vis viva* i.e., of double the kinetic energy, extended over the entire motion from P_1 to P_2. This time integral is called "action." It has a definite value for our tentative path and likewise for any other tentative path, these paths being always drawn between the same two end-points P_1, P_2 and always traversed with the same given energy constant E.

The value of this "action" will vary from path to path. For some paths it will come out larger, for others smaller. Mathematically we can imagine that *all* possible paths have been tried. There must exist one definite path (at least if P_1 and P_2 are not too far apart) for which the action assumes a minimum value. The principle of least action asserts that *this particular path is the one chosen by nature as the actual path of motion.*

We have explained the operation of the principle for one single particle. It can be generalized, however, to any number of particles and any arbitrarily complicated mechanical system.

3. Hamilton's procedure. We encounter problems of mechanics for which the work function is a function not only of the position of the particle but also of the time. For such systems the law of the conservation of energy does not hold, and the principle of Euler and Lagrange is not applicable, but that of Hamilton is.

In Hamilton's procedure we again start with the given initial point P_1 and the given end-point P_2. But now we do not restrict the trial motion in any way. Not only can the path be chosen arbitrarily—save for natural continuity conditions—but also the motion in time is at our disposal. All that we require now is that our tentative motion shall start at the observed time t_1 of the actual motion and end at the observed time t_2. (This condition is not satisfied in the procedure of Euler-Lagrange, because there the energy theorem restricts the motion, and the time taken to go from P_1 to P_2 in the tentative motion will generally differ from the time taken in the actual motion.)

The characteristic quantity that we now use as the measure of action—there is unfortunately no standard name adopted for this quantity—is the time-integral of the *difference between the kinetic and potential energies.* The Hamiltonian formulation of the principle of least action asserts that *the actual motion realized*

in nature is that particular motion for which this action assumes its smallest value.

One can show that in the case of "conservative" systems, i.e. systems which satisfy the law of the conservation of energy, the principle of Euler-Lagrange is a consequence of Hamilton's principle, but the latter principle remains valid even for non-conservative systems.

4. The calculus of variations. The mathematical problem of minimizing an integral is dealt with in a special branch of the calculus, called "calculus of variations." The mathematical theory shows that our final results can be established without taking into account the infinity of tentatively possible paths. We can restrict our mathematical experiment to such paths as are ***infinitely near*** to the actual path. A tentative path which differs from the actual path in an arbitrary but still ***infinitesimal*** degree, is called a "variation" of the actual path, and the calculus of variations investigates the changes in the value of an integral caused by such infinitesimal variations of the path.

5. Comparison between the vectorial and the variational treatments of mechanics. The vectorial and the variational theories of mechanics are two different mathematical descriptions of the same realm of natural phenomena. Newton's theory bases everything on two fundamental vectors: "momentum" and "force"; the variational theory, founded by Euler and Lagrange, bases everything on two scalar quantities: "kinetic energy" and "work function." Apart from mathematical expediency, the question as to the equivalence of these two theories can be raised. In the case of *free* particles, i.e. particles whose motion is not restricted by given "constraints," the two forms of description lead to equivalent results. But for systems with constraints the analytical treatment is simpler and more economical. The given constraints are taken into account in a natural way by letting the system move along all the tentative paths in harmony with them. The vectorial treatment has to take account of the forces which maintain the constraints and has to make definite hypotheses concerning them. Newton's third law of motion, "action equals reaction," does not embrace all cases. It suffices only for the dynamics of rigid bodies.

On the other hand, Newton's approach does not restrict the nature of a force, while the variational approach assumes that the acting forces are derivable from a scalar quantity, the "work function." Forces of a frictional nature, which have no work function, are outside the realm of variational principles, while the Newtonian scheme has no difficulty in including them.

Such forces originate from inter-molecular phenomena which are neglected in the macroscopic description of motion. If the macroscopic parameters of a mechanical system are completed by the addition of microscopic parameters, forces not derivable from a work function would in all probability not occur.

6. Mathematical evaluation of the variational principles. Many elementary problems of physics and engineering are solvable by vectorial mechanics and do not require the application of variational methods. But in all more complicated problems the superiority of the variational treatment becomes conspicuous. This superiority is due to the complete freedom we have in choosing the appropriate coordinates for our problem. The problems which are well suited to the vectorial treatment are essentially those which can be handled with a rectangular frame of reference, since the decomposition of vectors in curvilinear coordinates is a cumbersome procedure if not guided by the advanced principles of tensor calculus. Although the fundamental importance of invariants and covariants for all phenomena of nature has been discovered only recently and so was not known in the time of Euler and Lagrange, the variational approach to mechanics happened to anticipate this development by satisfying the principle of invariance automatically. We are allowed sovereign freedom in choosing our coordinates, since our processes and resulting equations remain valid for an arbitrary choice of coordinates. The mathematical and philosophical value of the variational method is firmly anchored in this freedom of choice and the corresponding freedom of arbitrary coordinate transformations. It greatly facilitates the formulation of the differential equations of motion, and likewise their solution. If we hit on a certain type of coordinates, called "cyclic" or "ignorable," a partial integration of the basic differential equations is at once accomplished. If *all* our coordinates are ignorable, our problem is completely solved. Hence, we can formulate the entire problem of solving the differential equations of motion as a problem of coordinate transformation. Instead of trying to integrate the differential equations of motion directly, we try to produce more and more ignorable coordinates. In the Euler-Lagrangian form of mechanics it is more or less accidental if we hit on the right coordinates, because we have no systematic way of producing ignorable coordinates. But the later developments of the theory by Hamilton and Jacobi broadened the original procedures immensely by introducing the "canonical equations," with their much wider transformation properties. Here we are able to produce a *complete* set of ignorable coordinates by solving one single partial differential equation.

Although the actual solution of this differential equation is possible only for a restricted class of problems, it so happens that many important problems of theoretical physics belong precisely to this class. And thus the most advanced form of analytical mechanics turns out to be not only esthetically and logically most satisfactory, but at the same time very practical by providing a tool for the solution of many dynamical problems which are not accessible to elementary methods.[1]

7. Philosophical evaluation of the variational approach to mechanics. Although it is tacitly agreed nowadays that scientific treatises should avoid philosophical discussions, in the case of the variational principles of mechanics an exception to the rule may be tolerated, partly because these principles are rooted in a century which was philosophically oriented to a very high degree, and partly because the variational method has often been the focus of philosophical controversies and misinterpretations.

[1]The present book does not discuss other integration methods which are not based on the transformation theory. Concerning such methods the reader is referred to the advanced text-books mentioned in the Bibliography.

Indeed, the idea of enlarging reality by including "tentative" possibilities and then selecting one of these by the condition that it minimizes a certain quantity, seems to bring a *purpose* to the flow of natural events. This is in contradiction to the usual *causal* description of things. Yet we must not be surprised that for the more universal approach which was current in the 17th and 18th centuries, the two forms of thinking did not necessarily appear contradictory. The keynote of that entire period was the seemingly pre-established harmony between "reason" and "world." The great discoveries of higher mathematics and their immediate application to nature imbued the philosophers of those days with an unbounded confidence in the fundamentally intellectual structure of the world. Thus "deus intellectualis" was the basic theme of the philosophy of Leibniz, no less than that of Spinoza. At the same time Leibniz had strong teleological tendencies, and his activities had no small influence on the development of variational methods.[1] But this is not surprising if we observe how the purely esthetic and logical interest in maximum-minimum problems gave one of the strongest impulses to the development of infinitesimal calculus, and how Fermat's derivation of the laws of geometrical optics on the basis of his "principle of quickest arrival" could not fail to impress the philosophically-oriented scientists of those days. That the dilettante misuse of these tendencies by Maupertuis and others for theological purposes has put the entire trend into disrepute, is not the fault of the great philosophers.

The sober, practical, matter-of-fact nineteenth century—which carries over into our day—suspected all speculative and interpretative tendencies as "metaphysical" and limited its programme to the pure description of natural events. In this philosophy mathematics plays the role of a shorthand method, a conveniently economical language for expressing involved relations. Hence, it is not surprising, but quite consistent with the "positivistic" spirit of the nineteenth century, to meet with the following appraisal of analytical mechanics by one of the leading

[1] See the attractive historical study of A. Kneser, *Das Prinzip der kleinsten Wirkung von Leibniz bis zur Gegenwart* (Leipzig: Teubner, 1928).

figures of that trend, E. Mach, in "The Science of Mechanics" (*Open Court*, 1893, p. 480): "No fundamental light can be expected from this branch of mechanics. On the contrary, the discovery of matters of principle must be substantially completed before we can think of framing analytical mechanics the sole aim of which is a perfect *practical* mastery of problems. Whosoever mistakes this situation will never comprehend Lagrange's great performance, which here too is essentially of an *economical* character." (Italics in the original.) According to this philosophy the variational principles of mechanics are not more than alternative mathematical formulations of the fundamental laws of Newton, without any primary importance.

However, philosophical trends float back and forth and the last word is never spoken. In our own day we have witnessed at least *one* fundamental discovery of unprecedented magnitude, namely Einstein's Theory of General Relativity, which was obtained by mathematical and philosophical speculation of the highest order. Here was a discovery made by a kind of reasoning that a positivist cannot fail to call "metaphysical," and yet it provided an insight into the heart of things that mere experimentation and sober registration of facts could never have revealed. The Theory of General Relativity brought once again to the fore the spirit of the great cosmic theorists of Greece and the eighteenth century.

In the light of the discoveries of relativity, the variational foundation of mechanics deserves more than purely formalistic appraisal. Far from being nothing but an alternative formulation of the Newtonian laws of motion, the following points suggest the supremacy of the variational method:

1. The Principle of Relativity requires that the laws of nature shall be formulated in an "invariant" fashion, i.e. independently of any special frame of reference. The methods of the calculus of variations automatically satisfy this principle, because the minimum of a scalar quantity does not depend on the coordinates in which that quantity is measured. While the Newtonian equations of motion did not satisfy the principle of relativity, the principle of least action remained valid, with the

only modification that the basic action quantity had to be brought into harmony with the requirement of invariance.

2. The Theory of General Relativity has shown that matter cannot be separated from field and is in fact an outgrowth of the field. Hence, the basic equations of physics must be formulated as partial rather than ordinary differential equations. While Newton's particle picture can hardly be brought into harmony with the field concept, the variational methods are not restricted to the mechanics of particles but can be extended to the mechanics of continua.

3. The Principle of General Relativity is automatically satisfied if the fundamental "action" of the variational principle is chosen as an invariant under any coordinate transformation. Since the differential geometry of Riemann furnishes us such invariants, we have no difficulty in setting up the required field equations. Apart from this, our present knowledge of mathematics does not give us any clue to the formulation of a co-variant, and at the same time consistent, system of field equations. Hence, in the light of relativity the application of the calculus of variations to the laws of nature assumes more than accidental significance.

CONTENTS

INTRODUCTION

I. THE BASIC CONCEPTS OF ANALYTICAL MECHANICS

II. THE CALCULUS OF VARIATIONS