Ten-Division Influence Lines for Continuous Beams

By

Dr.-Ing. GEORG ANGER

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Translated from the German by

CHARLES J. HYMAN

Ordinates of Influence Lines
and of Moment Curves for Continuous Beams

Influence Coefficients of Cantilever Moments

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CONTENTS

Three-moment equation, load coefficients, formulas for rapid computation of support moments (span moments and shearing forces) in cases of various span loads and span lengths for 2, 3, 4 and more spans

The three-moment equation[1]

Beams on more than two supports are statically indeterminate constructions. The statically indeterminate quantities are the moments on the supports. The moments at the initial and end supports are usually taken as 0 (simple support) or as the moments of a fixed beam if we have a right to assume a partial or complete restraint.

The calculation of support moments is achieved through Clapeyron's equation.[2]

If we denote by

$l_1, l_2 \ldots l_{(n-1)}, l_n, l_{(n-1)} \ldots$ the lengths of the spans,

$M_0, M_1 \ldots M_{\overline{(n}-1)}, M_{\overline{n}}, M_{\overline{(n}+1)}$ the moments on the supports,

$\mathfrak{M}_x^1, \mathfrak{M}_x^2 \ldots \mathfrak{M}_x^{(n-1)}, \mathfrak{M}_x^n, \mathfrak{M}_x^{(n+1)}$ the moments for x at the spans l_1 to $l_{(n+1)}$,

$y_0, y_1 \ldots y_{(n-1)}, y_n, y_{(n+1)}$ the sag of the supports resulting from the yielding of the abutments,

$A_0, A_1 \ldots A_{(n-1)}, A_n, A_{(n-1)}$ the reactions,

E the modulus of elasticity of the material,

J the moment of inertia of the beam,

h the height of the beam,

$t_u - t_o$ the temperature differential between the top and bottom cross-section fibers,

a the coefficient of expansion;

furthermore, if we imagine the continuous beam as cut up at the supports, then for the statically determinate beams on two supports thus obtained, having spans $l_1, l_2 \ldots l_{(n+1)}$ we denote by

$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \ldots$ the left (negative) reactions of the spans $l_1, l_2, l_3 \ldots$,

$\mathfrak{B}_1, \mathfrak{B}_2 \ldots$ the right (positive) reactions of the spans $l_1, l_2 \ldots$,

${}^0\mathfrak{M}_x^1, {}^0\mathfrak{M}_x^2 \ldots$ the moments for x in the spans $l_1, l_2 \ldots$,

$\mathfrak{Q}_x^1, \mathfrak{Q}_x^2 \ldots$ the shears for x in the spans $l_1, l_2 \ldots$,

$\mathfrak{F}_1, \mathfrak{F}_2 \ldots$ the contents of the simple moment areas of the spans $l_1, l_2 \ldots$

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[1] Known as *Clapeyron's equation;* in this general form, however, it is due to German statisticians and (according to *E. Chwalla*) was first published by *Bertot*.

[2] The derivation of *Clapeyron's theorem* from the fact that the line of flexure in its passage over a support exhibits, at the point of support on either side, the same support angle of rotation as vertex angle, is developed in detail in volume I. This rotation angle is equal to the reaction of a simple beam which is loaded by the moment area divided by *E.J*.

$6\,^{\mathfrak{M}}\mathfrak{A}_n$ six times the reaction of the simple moment area $(\mathfrak{F}_1, \mathfrak{F}_2) \ldots \mathfrak{F}_n$ at the left supports $0 \ldots (\bar{n}-1)$

$6\,^{\mathfrak{M}}\mathfrak{B}_n$ six times the reaction of the simple moment area $(\mathfrak{F}_1, \mathfrak{F}_2) \ldots \mathfrak{F}_n$ at the right supports I .... $(\bar{n})$ [3]

$6\,^{\mathfrak{M}}\mathfrak{A}_2$ six times the reaction of the simple moment area $\mathfrak{F}_2$ at the left support I

$6\,^{\mathfrak{M}}\mathfrak{B}_2$ six times the reaction of the moment area $\mathfrak{F}_2$ at the right support II;

then, by Clapeyron's equation we have for constant *J, E, h,* $(t_u - t_o)$:

$$\underline{M_{(\bar{n}-1)}\, l_n + 2\, M_{\bar{n}}\, (l_n + l_{(n+1)}) + M_{(\bar{n}+1)}\, l_{(n+1)} =}$$

$$\underline{= -\,6\, (^{\mathfrak{M}}\mathfrak{B}_n + {}^{\mathfrak{M}}\mathfrak{A}_{(n+1)}) + 6\, EJ \left(\frac{y_n - y_{(n+1)}}{l_{(n-1)}} + \frac{y_n - y_{(n-1)}}{l_n} \right) -}$$

$$\underline{-\, 3\, a\, EJ\, (t_u - t_o)\, \frac{l_n + l_{(n+1)}}{h}}. \tag{1}$$

When the supports are unyielding and the temperature differences are neglected we get $y_n = y_{(n+1)} = y_{(n-1)}$ and $t_u = t_o = 0$, and (1) becomes

$$\underline{M_{(\bar{n}-1)}\, l_n + 2\, M_{\bar{n}}\, (l_n + l_{(n+1)}) + M_{(\bar{n}+1)}\, l_{(n+1)} = -\,6 (^{\mathfrak{M}}\mathfrak{B}_2 + {}^{\mathfrak{M}}\mathfrak{A}_{(n+1)})} \tag{2}$$

The left side of (2) contains the statically indeterminate quantities $M_{(\bar{n}-1)}, M_{\bar{n}}, M_{(\bar{n}+1)}$ of three consecutive supports as well as the lengths of two intervening spans (members); the right side depends on the loading and the lengths of these two spans.

We can thus set up $(n-2)$ equations for n spans, and the two missing equations come from the characterization of the end supports, whose moments in the case of simple supports are to be put equal to zero.

From the $(n-2)$ equations we can (best by elimination) compute the n support moments as well as derive formulas for their calculation.

In the tables that follow, there will be found the values of $6\,^{\mathfrak{M}}\mathfrak{B}_n$ and $6\,^{\mathfrak{M}}\mathfrak{A}_{(n+1)}$ for the loadings most frequently used.

In conjunction therewith we have entered the uniform substitute load g_E which produces the support moment of the same magnitude, in order to make possible the use of the simpler formulas for uniform loads $\left[6\,^{\mathfrak{M}}\mathfrak{A} \text{ bzw. } 6\,^{\mathfrak{M}}\mathfrak{B} = \frac{g_E l^3}{4} \right]$.

The equal load coefficients g_E are serviceable only for the computation of support moments.

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[3]The negative support moments are denoted by the Latin letters *M*, the positive span moments by the German letters $\mathfrak{M}$, the span sequence by the Arabic numerals 1, 2, 3, ... *n*, the support sequence by the Roman numerals I, II, ... $\bar{n}$.

In the formulas, $(n-1)$, n, $(n+1)$ correspond to the Arabic numerals; $(\bar{n}-1)$, $\bar{n}$, $(\bar{n}+1)$ to the Roman numerals.

They can be used in the case of symmetric loading, where the reactions are equal on the left and right, in end spans as well as in central spans; in loadings, however, where unequal reactions occur, they can be used only in end spans in the case of hinged ends.

The values $6\,^{\mathfrak{M}}\mathfrak{A}$ and $6\,^{\mathfrak{M}}\mathfrak{B}$ are given in volume I for 53 distinct loading cases, with numerous subtables for load positions or load ends at the tenth- and twentieth-points.

In the following tables of load coefficients[4] only 17 of the more important loading cases are given; for the other cases, the reader is referred to volume I.

Concept and origin of support moments

Every beam resting on more than two supports (continuous beam) is, for calculation purposes, a beam on two supports whose span length is equal to that of all the spans of the continuous beam.

The middle supports are point loads acting on the beam from below; their magnitude is at first unknown, but is determined numerically through the fact that its lifting effect on the beam at the point of application of the load is equal to the sag caused by the actual loading from above.

These point loads, unknown in magnitude at first, are the **reactions of the middle supports.**

Because the moments on a single-span beam, which are produced by these reactions and tend upward (negative), are (as a rule) greater at the support points than the downward (positive) bending moments of the factual load, there arises in the superposition of the negative upon the positive moment areas an excess by way of negative bending moments (tending upward), which are called **support moments.**

In the center of the spans, on superposition of the negative upon the positive moment areas, the positive moment areas usually preponderate; the latter are called **span moments.**

However, there can also occur positive support moments and negative span moments in certain cases of loading.

Thus span moments and support moments are the difference of two moments acting on a one-span beam, where the positive are produced by the incident loading, the negative by the reactions.

In other words: where the continuous beam bends upward we have an upward pull, and we then speak of negative moments; where the continuous beam bends downward we have a downward pull, and we then speak of positive moments.

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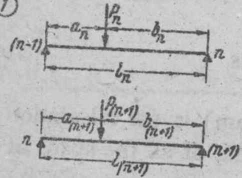
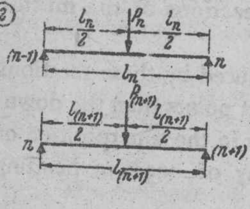
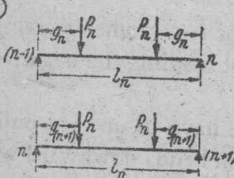
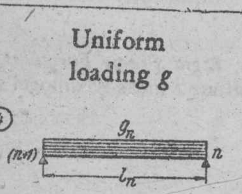
[4]***Translator's note:*** **Sometimes called "load terms." Cf. A. Kleinlogel, *Rigid Frame Formulas,* Frederick Ungar Publishing Co., New York, 1952. Besides, our "coefficients" are *l* times Kleinlogel's "terms."**

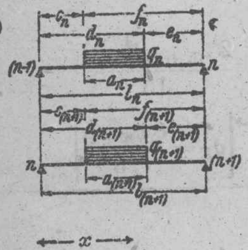
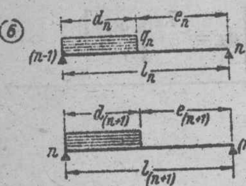
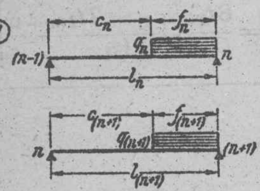
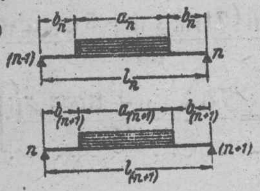
Load coefficients

Values for $6\,\mathfrak{M}\mathfrak{B}_n$ and $6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$ for different loads, equal load coefficients g_E

No. Type of load	$6\,\mathfrak{M}\mathfrak{B}_n$	$6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$
Load P concentrated at an arbitrary point ① 	$P\frac{a\,b}{l}(l+a)$ or $\frac{P_n\,a_n\left(l_n^2-a_n^2\right)}{l_n}$ $\left[g_E=\frac{4\,P_n\,a_n\left(l_n^2-a_n^2\right)}{l_n^4}\right]$	$P\frac{a\,b}{l}(l+b)$ or $\frac{P_{(n+1)}\,b_{(n+1)}\left(l_{(n+1)}^2-b_{(n+1)}^2\right)}{l_{(n+1)}}$ $\left[g_E=\frac{4\,P_{(n+1)}\,b_{(n+1)}\left(l_{(n+1)}^2-b_{(n+1)}^2\right)}{l_{(n+1)}^4}\right]$
Load P concentrated at center of span ② 	$\frac{3}{8}P_n\,l_n^2$ $\left[g_E=1{,}5\,\frac{P_n}{l_n}\right]$	$\frac{3}{8}P_{(n+1)}\,l_{(n+1)}^2$ $\left[g_E=1{,}5\,\frac{P_{(n+1)}}{l_{(n+1)}}\right]$
2 symmetrical single point loads P ③ 	$3\,P_n\,g_n\,(l_n-g_n)$ $\left[g_E=\frac{12\,P_n\,g_n\,(l_n-g_n)}{l_n^3}\right]$	$3\,P_{(n+1)}\,g_{(n+1)}\,(l_{(n+1)}-g_{(n+1)})$ $\left[g_E=\frac{12\,P_{(n+1)}\,g_{(n+1)}\,l_{(n+1)}-g_{(n+1)})}{l_{(n+1)}^3}\right]$
Uniform loading g ④ 	$\frac{g_n\,l_n^3}{4}$	$\frac{g_{(n+1)}\,l_{(n+1)}^3}{4}$

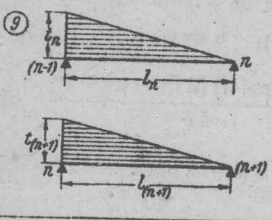
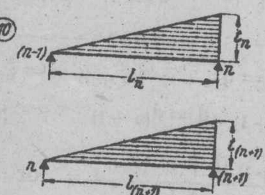
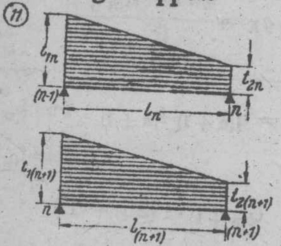
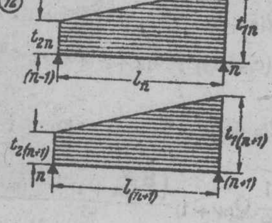
Load coefficients

Values for $6\,\mathfrak{M}\mathfrak{B}_n$ and $6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$ for different loads, equal load coefficients g_E

No.	Type of load	$6\,\mathfrak{M}\mathfrak{B}_n$	$6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$
5	Uniform load q  Note: $\mathfrak{M}_{max}$ occurs in x	$Q_n = q_n \cdot a_n$; $a = a_n$ $l = l_n$; $c = c_n$; $d = d_n$; $x = x_n$ $\frac{Q_n (c+d)(2l^2 - c^2 - d^2)}{4l}$ or $\frac{Q_n \cdot 2x \cdot (2l^2 - 2x^2 - 0{,}5a^2)}{4l}$ $\left[g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{B}_n}{l_n^3}\right]$	$Q_{(n+1)} = q_{(n+1)}\, a_{(n+1)}$ $a = a_{(n+1)}$ $l = l_{(n+1)}$ $e = e_{(n+1)}$ $f = f_{(n+1)}$ $\frac{Q_{(n+1)} (e+f)(2l^2 - e^2 - f^2)}{4l}$ or $\frac{Q_{(n+1)} \cdot 2(l-x)[4lx - 2x^2 - 0{,}5a^2]}{4l}$ $\left[g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{A}_{(n+1)}}{l_{(n+1)}^3}\right]$
6	Uniform load q at left support  Note: $\mathfrak{M}_{max}$ occurs in x	$Q_n = q_n \cdot d_n$ $l = l_n$; $d = d_n$ $x = x_n$ $\frac{Q_n d (2l^2 - d^2)}{4 \cdot l}$ or $\frac{Q_n \cdot 2x(2l^2 - d^2)}{4 \cdot l}$ $\left[g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{B}_n}{l_n^3}\right]$	$Q_{(n+1)} = q_{(n+1)} \cdot d_{(n+1)}$ $l = l_{(n+1)}$ $e = e_{(n+1)}$ $d = d_{(n+1)}$ $x = x_{(n+1)}$ $\frac{Q_{(n+1)} (l+e)(l^2 - e^2)}{4l}$ or $\frac{Q_{(n+1)} \cdot 2(l-x)[l^2 - (l-e)^2]}{4l}$ $g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{A}_{(n+1)}}{l_{(n+1)}^3}$
7	Uniform load q at right support 	$Q_n = q_n f_n$ $l = l_n$; $c = c_n$ $\frac{Q_n (l+c)\, l^2 - c^2)}{4l}$ $\left[g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{B}_n}{l_n^3}\right]$	$Q_{(n+1)} = q_{(n+1)}\, f_{(n+1)}$ $l = l_{(n+1)}$ $f = f_{(n+1)}$ $\frac{Q_{(n+1)} \cdot f(2l^2 - f^2)}{l}$ $g_E = \frac{4 \cdot 6\,\mathfrak{M}\mathfrak{A}_{(n+1)}}{l_{(n+1)}^3}$
8	Uniform load q symmetrically located 	$[Q_n = q_n\, a_n]$ $Q(c_n + d_n)\left(2l_n^2 - c_n^2 - d_n^2\right)$ $\frac{1}{8} Q_n \left(3l_n^2 - a_n^2\right)$ $\left[g_E = \frac{1}{2} Q_n \frac{\left(3l_n^2 - a_n^2\right)}{l_n^3}\right]$	$[Q_{(n+1)} = q_{(n+1)}\, a_{(n+1)}]$ $Q_{(n+1)}(c_{(n+1)} + d_{(n+1)})(2l^2 - c^2 - c)$ $\frac{1}{8} Q_{(n+1)} \left(3l_{(n+1)}^2 - a_{(n+1)}^2\right)$ $\left[g_E = \frac{1}{8} Q_{(n+1)} \frac{\left(3l_{(n+1)}^2 - a_{(n+1)}^2\right)}{l_{(n+1)}^3}\right]$

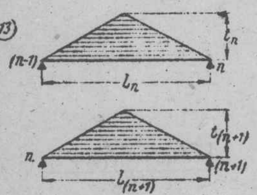
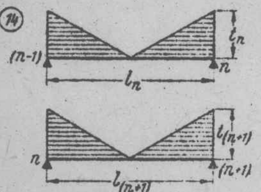
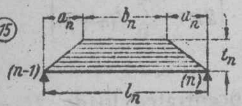
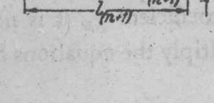
Load coefficients

Values for $6\,{}^{\mathfrak{M}}\mathfrak{B}_n$ and $6\,{}^{\mathfrak{M}}\mathfrak{A}_{(n+1)}$ for different loads, equal load coefficients g_E

No. Type of load	$6\,{}^{\mathfrak{M}}\mathfrak{B}_n$	$6\,{}^{\mathfrak{M}}\mathfrak{A}_{(n+1)}$
Triangular load with maximum ordinate t at left support ⑨ 	$\frac{7\,t_n\,l_n^3}{60}$ $\left[g_E = \frac{7\,t_n}{15}\right]$	$\frac{2\,t_{(n+1)}\,l_{(n+1)}^3}{15}$ $\left[g_E = \frac{8\,t_{(n+1)}}{15}\right]$
Triangular load with maximum ordinate t at right support ⑩ 	$\frac{2\,t_n\,l_n^3}{15}$ $\left[g_E = \frac{8\,t_n}{15}\right]$	$\frac{7\,t_{(n+1)}\,l_{(n+1)}^3}{60}$ $\left[g_E = \frac{7\,t_{(n+1)}}{15}\right]$
Trapezoidal load larger load ordinate t_1 at the left, smaller ordinate t_2 at the right support ⑪ 	$\frac{l_n^3\,(7\,t_{1n} + 8\,t_{2n})}{60}$ $\left[g_E = \frac{7\,t_{1n} + 8\,t_{2n}}{15}\right]$	$\frac{l_{(n+1)}^3\,(8\,t_{1(n+1)} + 7\,t_{2(n+1)})}{60}$ $\left[g_E = \frac{8\,t_{1(n+1)} + 7\,t_{2(n+1)}}{15}\right]$
Trapezoidal load larger load ordinate t_1 at the right, smaller ordinate t_2 at the left support ⑫ 	$\frac{l_n^3\,(8\,t_{1n} + 7\,t_{2n})}{60}$ $\left[g_E = \frac{8\,t_{1n} + 7\,t_{2n}}{15}\right]$	$\frac{l_{(n+1)}^3\,(7\,t_{1(n+1)} + 8\,t_{2(n+1)})}{60}$ $\left[g_E = \frac{7\,t_{1(n+1)} + 8\,t_{2(n+1)}}{15}\right]$

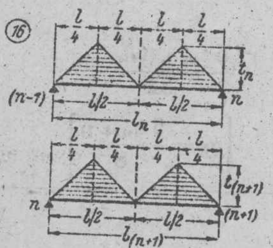
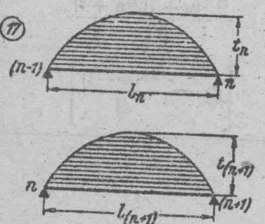
Load coefficients

Values for $6\,\mathfrak{M}\mathfrak{B}_n$ and $6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$ for different loads, equal load coefficients g_E

No. Type of load	$6\,\mathfrak{M}\mathfrak{B}_n$	$6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$
Triangular load with maximum ordinate t at the center (13) 	$\frac{5\,t_n\,l_n^3}{32}$ $\left[g_E = \frac{5\,t_n}{8}\right]$	$\frac{5\,t_{(n+1)}\,l_{(n+1)}^3}{32}$ $\left[g_E = \frac{5\,t_{(n+1)}}{8}\right]$
Triangular loads with maximum ordinate t at the supports and 0 in the center of the span (14) 	$\frac{3\,t_n\,l_n^3}{32}$ $\left[g_E = \frac{3\,t_n}{8}\right]$	$\frac{3\,t_{(n+1)}\,l_{(n+1)}^3}{32}$ $\left[g_E = \frac{3\,t_{(n+1)}}{8}\right]$
Symmetrical trapezoidal load (15) 	$\frac{t_n\,l_n^2\,(l_n + b_n)}{32}\left(5 + \frac{b_n^2}{l_n^2}\right)$ $\left[g_E = \frac{t_n\,(l_n + b_n)}{8\,l_n}\left(5 + \frac{b_n^2}{l_n^2}\right)\right]$	
	Special cases:	
	$\frac{b}{l} + \frac{2}{3}$; $6\,\mathfrak{M}\mathfrak{B}_n = 6\,\mathfrak{M}\mathfrak{A}_{(n+1)} = 0{,}284\ t_n\,l_n^3$ $\frac{3}{5}$ $= 0{,}268$ „ $\frac{1}{2}$ $= 0{,}246$ „ $\frac{1}{3}$ $= 0{,}213$ „ $\frac{1}{5}$ $= 0{,}189$ „	
		$\frac{t_{(n+1)}\,l_{(n+1)}^2\,(l_{(n+1)} + b_{(n+1)})}{32}\left(5 + \frac{b_{(n+1)}^2}{l_{(n+1)}^2}\right)$ $\left[g_E = \frac{t_{(n+1)}\,(l_{(n+1)} + b_{(n+1)})}{8\,l_{(n+1)}}\left(5 + \frac{b_{(n+1)}^2}{l_{(n+1)}^2}\right)\right]$

Load coefficients

Values for $6\,\mathfrak{M}\mathfrak{B}_n$ and $6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$ for different loads, equal load coefficients g_E

No. Type of load	$6\,\mathfrak{M}\mathfrak{B}_n$	$6\,\mathfrak{M}\mathfrak{A}_{(n+1)}$
2 Symmetrical triangular loads maximum ordinate t (16) 	$\frac{17\,t_n\,l_n^3}{128}$ $\left[g_E = \frac{17}{32}\,t_n\right]$	$\frac{17\,t_{(n+1)}\,l_{(n+1)}^3}{128}$ $\left[g_E = \frac{17}{32}\,t_{(n+1)}\right]$
Parabolic load with maximum ordinate t at center of parabola (17) 	$\frac{t_n\,l_n^3}{5}$ $\left[g_E = \frac{4}{5}\,t_n\right]$	$\frac{t_{(n+1)}\,l_{(n+1)}^3}{5}$ $\left[g_E = \frac{4}{5}\,t_{(n+1)}\right]$

For equal spans $l_n = l_{(n+1)}$ and equal symmetrically-located concentrated loads $P_n = P_{(n+1)}$ for equal uniform loads, symmetrically-located uniform loads of equal magnitude, and triangular as well as trapezoidal loads the values 6 become $6\,[\mathfrak{M}\mathfrak{B}_n + \mathfrak{M}\mathfrak{A}_{(n+1)}]$

1. $P\,(l^2 - a^2)\,\frac{a}{l}$
2. $\frac{3}{4}\,P\,l^2$
3. $6\,P\,g\,(l - g)$
4. $\frac{g\,l^3}{2}$
5. $\frac{Q\,(c + d)\,(2\,l^2 - c^2 - d^2)}{2\,l}$
6. $\frac{Q\,d\,(2\,l^2 - d^2)}{2\,l}$
7. $\frac{Q\,(l + c)\,(l^2 - c^2)}{2\,l}$
8. $\frac{Q\,[2\,l^2 - b^2 - (l - b)^2)]}{2}$
9. $\frac{7}{30}\,t\,l^3$
10. $\frac{4}{15}\,t\,l^3$
11. $\frac{l^3}{30}\,(7\,t_1 + 8\,t_2)$
12. $\frac{l^3}{30}\,(8\,t_1 + 7\,t_2)$
13. $\frac{5}{16}\,t\,l^3$
14. $\frac{3}{16}\,t\,l^3$
15. $\frac{t\,l^2\,(l + b)}{16}\left(5 + \frac{b^2}{l^2}\right)$
16. $\frac{17}{64}\,t\,l^3$
17. $\frac{2}{5}\,t\,l^3$

Note: In order to obtain the equal load coefficient g_E it is necessary to multiply the equations by $\frac{4}{l^3}$.

Support Moments

As we know, the support moments can be computed immediately from Clapeyron's equation. The critical (maximal) values for the various support moments, however, do not occur in the case of one and the same loading. Consequently the support moments would have to be computed for each case of loading, whereas, in every instance, only one particular case is of interest. This inconvenience can be avoided by finding the general solution of Clapeyron's equation and grouping the values of the support moments separately for the loading of each individual span. In order to obtain the critical values, we need then merely add the values arising from the single loads on the spans in question. The solution of Clapeyron's equation for two, three and four spans follows.[5]

Denote by:

$^1M_{I}$, $^1M_{II}$, $^1M_{III}$ the moments on the supports I, II, III for load in span 1
$^2M_{I}$, $^2M_{II}$, $^2M_{III}$ the moments on the supports I, II, III for load in span 2
$^3M_{I}$, $^3M_{II}$, $^3M_{III}$ the moments on the supports I, II, III for load in span 3
$^4M_{I}$, $^4M_{II}$, $^4M_{III}$ the moments on the supports I, II, III for load in span 4

then, referring to the notation on p. 1 ff., we have

A. Beam on 3 supports—2 spans and l_1 and l_2

$$^1M_{I} = -\frac{1}{2(l_1+l_2)} \cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{3}$$

$$^2M_{I} = -\frac{1}{2(l_1+l_2)} \cdot 6\,\mathfrak{M}\mathfrak{A}_2 \tag{4}$$

If l_1 and l_2 then

$$^1M_{I} = -\frac{6\,\mathfrak{M}\mathfrak{B}_1}{4\,l} \quad \text{and} \quad ^2M_{1} = -\frac{6\,\mathfrak{M}\mathfrak{A}_2}{4\,l} \tag{5}$$

B. Beam on 4 supports—3 spans l_1, l_2 and l_3

$$^1M_{I} = -\frac{2(l_2+l_3)}{4(l_1+l_2)(l_2+l_3)-l_2^2} \cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{6}$$

$$^2M_{I} = \frac{l_2}{4(l_1+l_2)(l_2+l_3)-l_2^2}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2 \cdot \frac{2(l_2+l_3)}{l_2}\right] \tag{7}$$

$$^3M_{I} = \frac{l_2}{4(l_1+l_2)(l_2+l_3)-l_2^2} \cdot 6\,\mathfrak{M}\mathfrak{A}_3 \tag{8}$$

$$^1M_{II} = \frac{l_2}{4(l_1+l_2)(l_2+l_3)-l_2^2} \cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{9}$$

$$^2M_{II} = -\frac{2(l_1+l_2)}{4(l_1+l_2)(l_2+l_3)-l_2^2}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2 \cdot \frac{l_2}{2(l_1+l_2)}\right] \tag{10}$$

$$^3M_{II} = -\frac{2(l_1+l_2)}{4(l_1+l_2)(l_2+l_3)-l_2^2} \cdot 6\,\mathfrak{M}\mathfrak{A}_3 \tag{11}$$

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[5] See volume I for the formulas for beams up to and including ten spans, freely supported and fixed at the ends.

Special cases:

1. If the 3 spans are equal, that is $l_1 = l_2 = l_3$, then:

$$^1M_{\mathrm{I}} = -\frac{4}{15\,l}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{12}$$

$$^2M_{\mathrm{I}} = \frac{1}{15\,l}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 4\cdot 6\,\mathfrak{M}\mathfrak{A}_2\right] \tag{13}$$

$$^3M_{\mathrm{I}} = \frac{1}{15\,l}\cdot 6\,\mathfrak{M}\mathfrak{A}_3 \tag{14}$$

$$^1M_{\mathrm{II}} = \frac{1}{15\,l}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{15}$$

$$^2M_{\mathrm{II}} = -\frac{1}{15\,l}\left[4\cdot 6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\right] \tag{16}$$

$$^3M_{\mathrm{II}} = -\frac{4}{15\,l}\cdot 6\,\mathfrak{M}\mathfrak{A}_3 \tag{17}$$

2. If the spans are unequal and only uniformly distributed loads are applied — p_1 in span 1, p_2 in span 2, p_3 in span 3, then:

$$^1M_{\mathrm{I}} = -\frac{(l_2 + l_3)}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_1 l_2^3}{2} \tag{18}$$

$$^2M_{\mathrm{I}} = -\frac{l_2 + 2\,l_3}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_2 l_2^3}{4} \tag{19}$$

$$^3M_{\mathrm{I}} = \frac{l_2}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_3 l_3^3}{4} \tag{20}$$

$$^1M_{\mathrm{II}} = \frac{l_2}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_1 l_1^3}{4} \tag{21}$$

$$^2M_{\mathrm{II}} = -\frac{2\,l_1 + l_2}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_2 l_2^3}{2} \tag{22}$$

$$^3M_{\mathrm{II}} = -\frac{(l_1 + l_2)}{4\,(l_1 + l_2)\,(l_2 + l_3) - l_2^2}\cdot\frac{p_3 l_3^3}{2} \tag{23}$$

Note: The denominator $4\,(l_1 + l_2)(l_2 + l_3) - l_2^2$ is the same for all moment values.

3. If in the case of uniform loading we have also $l_1 = l_2 = l_3$, then the above equations become:

$$^1M_{\mathrm{I}} = -\frac{1}{15}\,p_1 l^2 \tag{24}$$

$$^2M_{\mathrm{I}} = -\frac{1}{20}\,p_2 l^2 \tag{25}$$

$$^3M_{\mathrm{I}} = \frac{1}{60}\,p_3 l^2 \tag{26}$$

$$^1M_{\mathrm{II}} = \frac{1}{60}\,p_1 l^2 \tag{27}$$

$$^2M_{\mathrm{II}} = -\frac{1}{20}\,p_2 l^2 \tag{28}$$

$$^3M_{\mathrm{II}} = -\frac{1}{15}\,p_3 l^2 \tag{29}$$

and for uniform loading applied simultaneously in all three spans (dead load)

$$M_{\mathrm{I}} = M_{\mathrm{II}} = -\frac{1}{10}\,p\,l^2 \tag{30}$$

4. If $l_1 = l_3$ (equal end spans) and $l_2 = n\,l_1$, then we have:

$$ {}^1M_{\mathrm{I}} = -\frac{p_1 l_1^2}{2}\cdot\frac{(1+n)}{4(1+n)^2-n^2} \tag{31}$$

$$ {}^2M_{\mathrm{I}} = -\frac{p_2 l_1^2}{4}\cdot\frac{(2+n)\,n^3}{4(1+n)^2-n^2} \tag{32}$$

$$ {}^3M_{\mathrm{I}} = \frac{p_3 l_1^2}{4}\cdot\frac{n}{4(1+n)^2-n^2} \tag{33}$$

$$ {}^1M_{\mathrm{II}} = \frac{p_1 l_1^2}{4}\cdot\frac{n}{4(1+n)^2-n^2} \tag{34}$$

$$ {}^2M_{\mathrm{II}} = -\frac{p_2 l_1^2}{4}\cdot\frac{(2+n)\,n^3}{4(1+n)^2-n^2} \tag{35}$$

$$ {}^3M_{\mathrm{II}} = -\frac{p_3 l_1^2}{2}\cdot\frac{(1+n)}{4(1+n)^2-n^2} \tag{36}$$

and for simultaneous uniform loading in all 3 spans (dead load)

$$ M_{\mathrm{I}} = M_{\mathrm{II}} = -\frac{(2+n)(1+n^3)}{4(1+n)^2-n^2}\cdot\frac{p\,l_1^2}{4} \tag{37}$$

C. Beam on 5 supports—4 spans l_1, l_2, l_3 and l_4

Denominator: $N = (l_3+l_4)\left[4(l_1+l_2)(l_2+l_3)-l_2^2\right]-(l_1+l_2)\,l_3^2$ (38)

$$ {}^1M_{\mathrm{I}} = -\frac{1}{2(l_1+l_2)}\cdot 6\,\mathfrak{M}\mathfrak{B}_1\left[\frac{l_2^2(l_3+l_4)}{N}+1\right] \tag{39}$$

$$ {}^2M_{\mathrm{I}} = -\frac{l_2(l_3+l_4)}{N}\left[6\,\mathfrak{M}\mathfrak{B}_2-6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{l_2}{2(l_1+l_2)}\right]-\frac{6\,\mathfrak{M}\mathfrak{A}_2^{\,2}}{l_2}\cdot\frac{1}{2(l_1+l_2)} \tag{40}$$

$$ {}^3M_{\mathrm{I}} = -\frac{l_2(l_3+l_4)}{N}\left[\frac{l_3}{2(l_3+l_4)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3-6\,\mathfrak{M}\mathfrak{A}_3\right] \tag{41}$$

$$ {}^4M_{\mathrm{I}} = -\frac{l_2 l_3}{2N}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 \tag{42}$$

$$ {}^1M_{\mathrm{II}} = \frac{l_2(l_3+l_4)}{N}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{43}$$

$$ {}^2M_{\mathrm{II}} = -\frac{2(l_1+l_2)(l_3+l_4)}{N}\left[6\,\mathfrak{M}\mathfrak{B}_2-6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{l_2}{2(l_1+l_2)}\right] \tag{44}$$

$$ {}^3M_{\mathrm{II}} = \frac{2(l_1+l_2)(l_3+l_4)}{N}\left[\frac{l_3}{2(l_3+l_4)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3-6\,\mathfrak{M}\mathfrak{A}_3\right] \tag{45}$$

$$ {}^4M_{\mathrm{II}} = \frac{l_3(l_1+l_2)}{N}\,6\,\mathfrak{M}\mathfrak{A}_4 \tag{46}$$

$$ {}^1M_{\mathrm{III}} = -\frac{l_2\, l_3}{2\,N}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{47} $$

$$ {}^2M_{\mathrm{III}} = \frac{l_3\,(l_1+l_2)}{N}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{l_2}{2\,(l_1+l_2)}\right] \tag{48} $$

$$ {}^3M_{\mathrm{III}} = -\frac{l_3\,(l_1+l_2)}{N}\left[\frac{l_3}{2\,(l_3+l_4)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3 - 6\,\mathfrak{M}\mathfrak{A}_3\right] - \frac{1}{2\,(l_3+l_4)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3 \tag{49} $$

$$ {}^4M_{\mathrm{III}} = -\frac{1}{2\,(l_3+l_4)}\cdot 6\,\mathfrak{M}\mathfrak{A}_4\left[\frac{l_3^{\,2}\,(l_1+l_2)}{N}+1\right] \tag{50} $$

Special cases:

1. If the 4 spans are equal $(l_1 = l_2 = l_3 = l_4)$., then we have:

$$ {}^1M_{\mathrm{I}} = -\frac{15}{56\,l}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{51} $$

$$ {}^2M_{\mathrm{I}} = \frac{1}{56\,l}\left[4\cdot 6\,\mathfrak{M}\mathfrak{B}_2 - 15\cdot 6\,\mathfrak{M}\mathfrak{A}_2\right] \tag{52} $$

$$ {}^3M_{\mathrm{I}} = \frac{1}{56\,l}\left[4\cdot 6\,\mathfrak{M}\mathfrak{A}_3 - 6\,\mathfrak{M}\mathfrak{B}_3\right] \tag{53} $$

$$ {}^4M_{\mathrm{I}} = -\frac{1}{56\,l}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 \tag{54} $$

$$ {}^1M_{\mathrm{II}} = \frac{1}{14\,l}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 = -4\cdot{}^1M_{\mathrm{III}} \tag{55} $$

$$ {}^2M_{\mathrm{II}} = \frac{1}{14\,l}\left[6\,\mathfrak{M}\mathfrak{A}_2 - 4\cdot 6\,\mathfrak{M}\mathfrak{B}_2\right] = -4\cdot{}^2M_{\mathrm{III}} \tag{56} $$

$$ {}^3M_{\mathrm{II}} = \frac{1}{14\,l}\left[6\,\mathfrak{M}\mathfrak{B}_3 - 4\cdot 6\,\mathfrak{M}\mathfrak{A}_3\right] = -\frac{4}{15}\,{}^3M_{\mathrm{I}} \tag{57} $$

$$ {}^4M_{\mathrm{II}} = \frac{1}{14\,l}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 = -\frac{4}{15}\cdot{}^4M_{\mathrm{I}} \tag{58} $$

$$ {}^1M_{\mathrm{III}} = -\frac{1}{56\,l}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{59} $$

$$ {}^2M_{\mathrm{III}} = \frac{1}{56\,l}\left[4\cdot 6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\right] \tag{60} $$

$$ {}^3M_{\mathrm{III}} = \frac{1}{56\,l}\left[4\cdot 6\,\mathfrak{M}\mathfrak{A}_3 - 15\cdot 6\,\mathfrak{M}\mathfrak{B}_3\right] \tag{61} $$

$$ {}^4M_{\mathrm{III}} = -\frac{15}{56\,l}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 \tag{62} $$

2. If $l_1 = l_4$ (equal end spans) and $l_2 = l_3 = n\,l_1$ (equal interior spans), then:

$$ {}^1M_{\mathrm{I}} = -\frac{1}{2\,(n+1)\,l_1}\cdot 6\,\mathfrak{M}\mathfrak{B}_1\left[\frac{n^2}{8\,n+6\,n^2}+1\right] \tag{63} $$

$$ {}^2M_{\mathrm{I}} = \frac{n}{l_1\,(8\,n+6\,n^2)}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{n}{(n+1)}\right] - 6\,\mathfrak{M}\mathfrak{A}_2\,\frac{1}{2\,(n+1)} \tag{64} $$

$$ {}^{3}M_{\mathrm{I}} = -\frac{n}{l_1(8n+6n^2)}\left[\frac{n}{2(1+n)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3 - 6\,\mathfrak{M}\mathfrak{A}_3\right] \tag{65} $$

$$ {}^{4}M_{\mathrm{I}} = -\frac{n}{l_1(12n^2+28n+16)}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 \tag{66} $$

$$ {}^{1}M_{\mathrm{II}} = \frac{n}{l_1(8n+6n^2)}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{67} $$

$$ {}^{2}M_{\mathrm{II}} = -\frac{2(n+1)}{l_1(8n+6n^2)}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{n}{2(1+n)}\right] \tag{68} $$

$$ {}^{3}M_{\mathrm{II}} = \frac{2(n+1)}{l_1(8n+6n^2)}\left[\frac{n}{2(1+n)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3 - 6\,\mathfrak{M}\mathfrak{A}_3\right] \tag{69} $$

$$ {}^{4}M_{\mathrm{II}} = \frac{n}{l_1(8n+6n^2)}\cdot 6\,\mathfrak{M}\mathfrak{A}_4 \tag{70} $$

$$ {}^{1}M_{\mathrm{III}} = -\frac{n}{l_1(12n^2+28n+16)}\cdot 6\,\mathfrak{M}\mathfrak{B}_1 \tag{71} $$

$$ {}^{2}M_{\mathrm{III}} = \frac{n}{l_1(8n+6n^2)}\left[6\,\mathfrak{M}\mathfrak{B}_2 - 6\,\mathfrak{M}\mathfrak{A}_2\cdot\frac{n}{2(1+n)}\right] \tag{72} $$

$$ {}^{3}M_{\mathrm{III}} = -\frac{n}{l_1(8n+6n^2)}\left[\frac{n}{2(1+n)}\cdot 6\,\mathfrak{M}\mathfrak{B}_3 - 6\,\mathfrak{M}\mathfrak{A}_3\right] - \frac{1}{2(1+n)l_1}\cdot 6\,\mathfrak{M}\mathfrak{A}_3 \tag{73} $$

$$ {}^{4}M_{\mathrm{III}} = -\frac{1}{2(n+1)l_1}\cdot 6\,\mathfrak{M}\mathfrak{A}_4\left[\frac{n^2}{8n+6n^2}+1\right] \tag{74} $$

3. If only uniformly distributed loads are applied, then in the case of unequal spans:

Denominator N (see equation 38, p. 11)

$$ {}^{1}M_{\mathrm{I}} = -\frac{p_1 l_1^3}{8(l_1+l_2)}\left[\frac{l_3^2(l_3+l_4)}{N}+1\right] \tag{75} $$

$$ {}^{2}M_{\mathrm{I}} = \frac{p_2 l_2^3}{4}\left[\frac{l_2(l_3+l_4)}{N}\left(1-\frac{l_2}{2(l_1+l_2)}\right)-\frac{1}{2(l_1+l_2)}\right] \tag{76} $$

$$ {}^{3}M_{\mathrm{I}} = -\frac{p_3 l_3^3}{4}\left(\frac{l_3}{2(l_3+l_4)}-1\right)\frac{l_2(l_3+l_4)}{N} \tag{77} $$

$$ {}^{4}M_{\mathrm{I}} = -\frac{p_4 l_4^3}{8}\cdot\frac{l_2 l_3}{N} \tag{78} $$

$$ {}^{1}M_{\mathrm{II}} = \frac{p_1 l_1^3}{4}\cdot\frac{l_2(l_3+l_4)}{N} \tag{79} $$

$$ {}^{2}M_{\mathrm{II}} = -\frac{p_2 l_2^3}{4}\left(1-\frac{l_2}{2(l_1+l_2)}\right)\frac{2(l_1+l_2)(l_3+l_4)}{N} \tag{80} $$

$$ {}^{3}M_{\mathrm{II}} = \frac{p_3 l_3^3}{4}\left(\frac{l_3}{2(l_3+l_4)}-1\right)\frac{2(l_1+l_2)(l_3+l_4)}{N} \tag{81} $$

$$ {}^{4}M_{\mathrm{II}} = \frac{p_4 l_4^3}{4}\cdot\frac{l_3(l_1+l_2)}{N} \tag{82} $$