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MICROMECHANICS SET

Volume 1

Micromechanics of Fracture and Damage

Luc Dormieux and Djimédo Kondo

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coordinated by
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Notations

$\underline{z}$:	position vector
z_i:	ith component of $\underline{z}$ in a cartesian orthonormal frame
$z = z_1 + iz_2$:	complex number associated with the position vector
$U(\underline{z})$:	Airy's stress function at point $\underline{z}$
$\phi(z)$, $\chi(z)$, $\psi(z) = \chi'(z)$:	complex potentials related to $U(\underline{z})$ by [1.17]
$\phi(\underline{z})$, $\psi(\underline{z})$:	Papkovitch–Neuber (real) potentials
$z = \omega(\zeta)$:	conformal map
$\boldsymbol{\sigma}(\underline{z})$:	stress tensor at point $\underline{z}$
$\boldsymbol{\varepsilon}(\underline{z})$:	strain tensor at point $\underline{z}$
$\underline{\xi}(\underline{z})$:	displacement vector at point $\underline{z}$
$\mathbf{G}(\underline{z})$:	second-order Green's tensor for an infinite elastic (2D or 3D) continuum
$\mathbb{C}$:	fourth-order stiffness tensor
$\mathbb{C}^s$:	solid stiffness tensor
$\mathbb{S}^s$:	solid compliance tensor
$\mathbb{C}^{\text{hom}}$:	macroscopic (homogenized or effective) stiffness tensor

k: elastic bulk modulus
μ: elastic shear modulus
ν: Poisson ratio
E: Young's modulus
k^s: elastic bulk modulus of the solid phase
μ^s: elastic shear modulus of the solid phase
ν^s: Poisson ratio of the solid phase
E^s: Young's modulus of the solid phase
$\mathbf{1}$: second-order unit tensor
$\mathbb{I}$: symmetric fourth-order unit tensor
$\mathbb{J}$: fourth-order spherical projector
$\mathbb{K}$: fourth-order deviatoric projector ($\mathbb{I} = \mathbb{J} + \mathbb{K}$)
$\mathbb{\Gamma}^{\infty}(\underline{z})$: Kernel of the Green operator (infinite space)
$\mathbb{\Gamma}(\underline{z}, \underline{z}')$: Kernel of the Green operator (finite domain)
Γ: Green operator
$\mathbf{\Sigma}$: macroscopic Cauchy stress tensor
$\mathbf{E}$: macroscopic strain tensor
$\mathbb{P}$: fourth-order Hill's tensor
J_o: zero-order Bessel function
$\mathcal{Y}(t)$: Heaviside function
$\mathcal{F}_s$: Fourier sine transform
$\mathcal{F}_c$: Fourier cosine transform
$\mathcal{H}$: Hankel transform
K_I: stress intensity factor (mode I)
K_{II}: stress intensity factor (mode II)
G: energy release rate
G_c: fracture energy
d: crack density parameter
$\mathcal{Y}_d$: driving force of damage propagation

Preface

And it shall come to pass, while my glory passeth by,
that I will put thee in a clift of the rock,
and will cover thee with my hand while I pass by.

Exodus 33:22

An examination of the literature devoted to cracked media reveals that there are two main options for the geometrical modeling of cracks:

– the first option [GRI 21] consists of the idealized representation of the crack as two parallel faces (segments in plane strain/stress conditions or plane surfaces in three dimensions [3D]). The usual approach consists of adopting stress free boundary conditions on the crack faces. The two faces asymptotically coincide in this mathematical idealization and the displacement undergoes a discontinuity across the crack line (respectively, surface). Indeed, the displacement vectors of two material points located on each face at the same geometrical point in the initial configuration can differ from one another. Clearly, the discontinuity of the displacement field is a consequence of the idealization of the crack as a geometrical entity having a measure equal to zero in the integral sense. For the same reason, the stresses at a crack tip are singular, which has led to the introduction of the well-known concept of stress intensity factors. This first model is referred to throughout the book as the Griffith crack model. It will be presented in two-dimensional conditions (plane strain/stress), as well as in 3D conditions;

– as a second option, the crack is represented as a flat cavity. For instance, it will be a flat ellipse in plane strain/stress conditions, or a flat spheroid (or ellipsoid) in 3D, characterized by an infinitesimal aspect ratio. Consequently, the mathematical measure (in the integral sense) of the crack is infinitesimal but non-zero. This point of view represents the cracked medium as a heterogeneous material and the crack itself as an inhomogeneity in the sense of the homogenization theory. This geometrical description will therefore be referred to as the inhomogeneity model. As long as the aspect ratio has a small but non-zero value, the latter model warrants the ability to define a continuous extension of the displacement field in the crack cavity, as done classically in micromechanics of porous media. It also avoids the occurrence of stress singularities.

The very existence of two geometrical models for the same physical entity raises the question of their consistency. As pointed out above, one model induces mathematical singularities while the other model preserves the continuity of the displacement field and the absence of stress singularity, provided that the aspect ratio remains infinitesimal but non-zero. This of course might erroneously suggest that the two models are not compatible. In fact, the consistency must be examined in an asymptotic sense, when the crack aspect ratio tends to 0. It will be shown that the two models yield perfectly consistent estimates in terms of effective elastic properties. A thorough comparison of the local stress, strain and displacement fields is also proposed.

The book is organized as follows:

– Chapter 1 presents some mathematical tools of the theory of linear elasticity, which will be useful in forthcoming developments. Beginning with plane elasticity, the method of the Airy function is recalled. Biharmonic stress functions can be generated in a systematic way by means of the complex potential approach, which is also briefly presented. The method of the Airy function will be implemented in the framework of each of the two geometrical models;

– in view of application to the inhomogeneity model, Chapter 2 first introduces the Green's function. This paves the way for a presentation of the so-called inclusion and inhomogeneity Eshelby problems. Indeed, the solutions of the latter requires the determination of the Hill tensor, which is defined from the derivatives of the Green's function. Eshelby's inclusion problem is a first step toward the concept of polarization. This motivates the introduction of the Green operator. These tools will be essential for the derivation of variational bounds on the effective elastic properties of microcracked media;

– Chapter 3 deals with the Griffith crack model in two-dimensional conditions. To begin with, the stress singularity at the crack tip and the stress intensity factors are introduced. Then, the complete solutions to mode I and mode II loadings are derived, based on the use of a displacement potential technique (Papkovitch–Neuber potential), which is directly presented in the context of its implementation to crack problems. This yields the corresponding stress intensity factors. Similarly, Chapter 5 deals with the Griffith crack model in 3D conditions. Again the complete 3D solutions in mode I and in shear mode are derived;

– Chapter 4 is devoted to the inhomogeneity model of crack in two-dimensional conditions. The cross-section of the crack is assumed to be a flat ellipse. Two different mathematical techniques are implemented, namely the complex potential approach of the Airy stress function and the solution to the Eshelby inhomogeneity problem. The same Eshelby-based technique is used in Chapter 6 in order to deal with 3D flat spheroidal cracks;

– Chapter 7 introduces the concept of energy release rate and presents the classical thermodynamic reasoning leading to the related criterion for crack propagation;

– the second part of the book is devoted to the effective properties of microcracked media and to damage modeling. It opens with Chapter 8, which proposes a brief introduction to the homogenization of heterogeneous elastic media. The two geometrical models for microcracks (Griffith crack and inhomogeneity model) are successively considered. These two routes are explored in Chapter 9 (Griffith crack) and in Chapters 10 and 11 (inhomogeneity model);

– Chapter 12 is devoted to the variational approach to effective properties. It first presents the energy-based definition of the effective stiffness. Then, the Hashin–Shtrikman–Willis variational approach is detailed. The discussion emphasizes the respective roles of the inhomogeneity shape (flat spheroid in the present case) and of the crack spatial distribution;

– eventually, a micromechanics-based damage constitutive law can be formulated and this is the aim of Chapter 13, which serves as a conclusion to this book. Uniqueness and stability issues concerning the damage model will be discussed.

Rock of Ages, cleft for me,
Let me hide myself in Thee.

Luc DORMIEUX
Djimédo KONDO
February 2016

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