Series of Congress Reports

Representation Theory – Current Trends and Perspectives

Henning Krause
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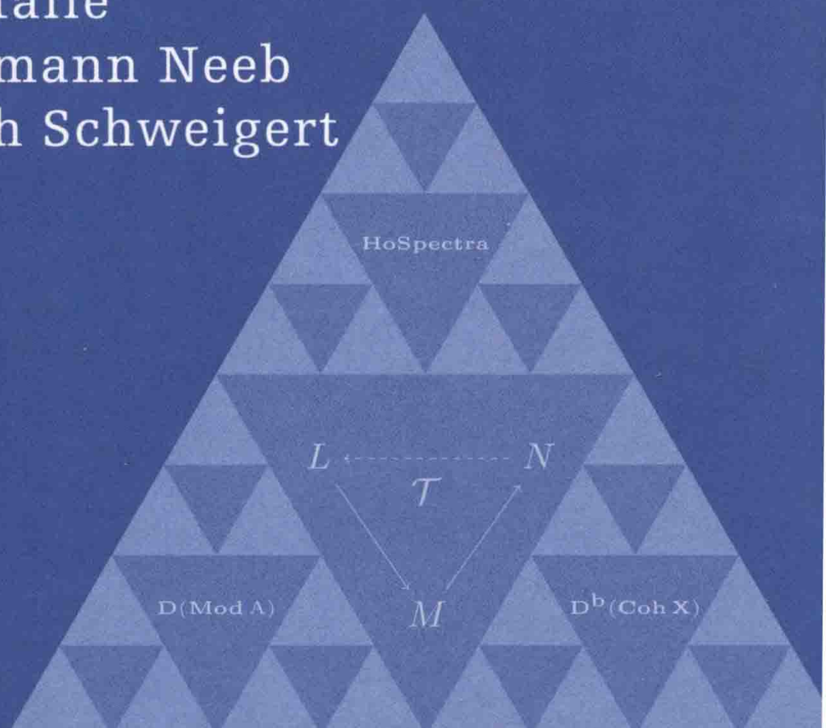
Editors



European Mathematical Society

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Preface

From April 2009 until March 2016, the German Science Foundation supported generously the Priority Program SPP 1388 in *Representation theory*. The core principles of the projects realized in the framework of the priority program have been *categorification* and *geometrization*, this is also reflected by the contributions to this volume. Another aim of the priority program was to strengthen the interaction between the different research areas in representation theory: from analytic representation theory, algebraic group theory over finite group theory up to the representation theory of finite dimensional algebras.

The proposal was submitted by Henning Krause (Bielefeld), Peter Littelmann (Cologne), Gunter Malle (Kaiserslautern), Karl-Hermann Neeb (Erlangen) and Christoph Schweigert (Hamburg).

In the framework of the priority program, the German Science Foundation financed many post-doc positions, permitting young representation theorists to pursue their own research objectives. We invited all of them to contribute to this proceedings volume.

Apart from the articles by former postdocs supported by the priority program, the volume contains a number of invited research and survey articles, many of them are extended versions of talks given at the last joint meeting of the priority program in Bad Honnef in March 2015.

The priority program benefitted in addition significantly from the excellent cooperation with representation theory networks in France and Great Britain (*Representation Theory Across The Channel*). We would like to use the opportunity to thank the responsible people of the networks in France and Great Britain for the smooth partnership, and the CNRS, DFG and EPSRC for allowing us to establish this cooperation with a rather limited amount of bureaucracy.

The editors thank the German Science Foundation for its support.

We give now a short survey of the content of the contributions:

Alexander Alldridge considers the properties of the restriction of invariant polynomials on the tangent space of a Riemannian symmetric supermanifold to a Cartan subspace. He gives a survey on the known results in the case the symmetric space is a Lie supergroup, and more generally, where the Cartan subspace is even.

Giovanni Cerulli Irelli reports on joint work with Markus Reineke and Evgeny Feigin on the geometry of quiver Grassmannians of Dynkin type with applications to cluster algebras. The paper contains several new proofs, in particular, a new proof of the positivity of cluster monomials in the acyclic clusters associated with Dynkin quivers is obtained.

The two papers by Stéphanie Cupit-Foutou and Guido Pezzini concern the theory of spherical varieties. Cupit-Foutou's article is a brief overview on recent classification results and related problems, Pezzini's article reviews some applications of the theory of spherical varieties in related fields, some generalizations of this theory, and he presents some open problems.

Olivier Dudas, Michela Varagnolo and Eric Vasserot explain in their contribution how Lusztig's induction and restriction functors yield categorical actions of Kac-Moody algebras on the derived category of unipotent representations. They focus on the example of finite general linear groups and induction/restriction associated with split Levi subgroups, providing a derived analog of Harish-Chandra induction/restriction as studied by Chuang-Rouquier.

Wolfgang Ebeling gives a survey on results related to the Berglund-Hübsch duality of invertible polynomials and the homological mirror symmetry conjecture for singularities.

Ben Elias, Noah Snyder and Geordie Williamson provide a diagrammatic description of some natural transformations between compositions of induction and restriction functors, in terms of colored transversely-intersecting planar 1-manifolds. The relations arise naturally in the work on (singular) Soergel bimodules.

Michael Ehrig and Catharina Stroppel study the combinatorics of the category $\mathcal{F}$ of finite dimensional integrable modules for the orthosymplectic Lie supergroup $OSp(r|2n)$. They present a positive counting formula for the dimension of the space of homomorphisms between two projective modules, refining earlier results of Gruson and Serganova. Moreover, they provide a direct link from $\mathcal{F}$ to the geometry of isotropic Grassmannians and Springer fibers of type $\mathtt{B}/\mathtt{D}$, and to parabolic categories $\mathcal{O}$ of type $\mathtt{B}/\mathtt{D}$, with maximal parabolic of type $\mathtt{A}$.

Xin Fang, Ghislain Fourier and Peter Littelmann provide a survey on T-equivariant toric degenerations of flag varieties. They explain how powerful tools in algebraic geometry and representation theory, such as canonical bases, Newton-Okounkov bodies, PBW-filtrations and cluster varieties come to push the subject forward, and discuss as application the determination of the Gromov width of flag varieties.

Peter Fiebig introduces in his article the subquotient categories of the restricted category $\mathcal{O}$ over an affine Kac–Moody algebra at the critical level and shows, that some of them have a realization in terms of moment graph sheaves.

Jürgen Fuchs and Christoph Schweigert show how structures in low-dimensional topology and low-dimensional geometry–often combined with ideas from (quantum) field theory–can explain and inspire concepts in algebra and in representation theory and their categorified versions.

Martin Kalck studies composition series of derived module categories. He shows that having a composition series with all factors being derived categories of vector spaces does not characterise derived categories of quasi-hereditary algebras. He also shows that derived categories of quasi-hereditary algebras can have composition series with lots of different lengths and composition factors. In other words, there is no Jordan-Hölder property for composition series of derived categories of quasi-hereditary algebras.

Steffen Koenig gives a survey on dominant dimension and its applications, guided by examples worked out in detail. Dominant dimension is a little known homological dimension, which is, however, crucial in many respects, both for abstractly studying finite dimensional algebras and their representation theory, and for applications to group algebras or in algebraic Lie theory. Various aspects and recent applications of dominant dimension are outlined and illustrated.

Henning Krause discusses in his contribution highest weight categories and strict polynomial functors. Highest weight categories are described in terms of standard objects and recollements of abelian categories, working over an arbitrary commutative base ring. The highest weight structure for categories of strict polynomial functors are explained, using the theory of Schur and Weyl functors. A consequence is the well-known fact that Schur algebras are quasi-hereditary.

Julian Külshammer gives a survey on bocses, quasi-hereditary algebras and their relationship. Particular emphasis is placed on applications of this result to the representation type of the category of modules of a quasi-hereditary algebra, which are filtered by standard modules.

Sefi Ladkani constructs a new class of symmetric algebras of tame representation type that are also the endomorphism algebras of cluster-tilting objects in 2-Calabi-Yau triangulated categories, hence all their non-projective indecomposable modules are Ω-periodic of period dividing 4. The construction may serve as a bridge between the modular representation theory of finite groups and the theory of cluster algebras.

Martina Lanini discusses some appearances of semi-infinite combinatorics in representation theory. She proposes a semi-infinite moment graph theory and motivates it by considering the geometric side of the story. She shows that it is possible to compute stalks of the local intersection cohomology of the semi-infinite flag variety, and hence of spaces of quasi maps, by performing an algorithm due to Braden and MacPherson.

Gunter Malle gives a survey of recent developments in the investigation of the various local-global conjectures for representations of finite groups. This article finds a perfect extension in the contribution by Britta Späth mentioned below.

Karl-Hermann Neeb describes in his contribution the recent progress in the classification of bounded and semibounded representations of infinite dimensional Lie groups. He starts with a discussion of the semiboundedness condition and discusses how the new concept of a smoothing operator can be used to construct C^*-algebras, thus making the full power of C^*-theory available in this context.

Ivan Penkov and Alexey Petukhov provide a review of results on two-sided ideals in the enveloping algebra $U(g(\infty))$ of a locally simple Lie algebra $g(\infty)$. They pay special attention to the case when $g(\infty)$ is one of the finitary Lie algebras $sl(\infty); o(\infty); sp(\infty)$.

Markus Reineke describes in his article the construction of small desingularizations of moduli spaces of semistable quiver representations for indivisible dimension vectors using deformations of stabilities and a dimension estimate for nullcones. He gives applications to the construction of several classes of GIT quotients.

Henrik Seppänen and Valdemar V. Tsanov study geometric invariant theory on a flag variety G/B with respect to the action of a principal 3-dimensional simple subgroup $S \subset G$. The GIT-quotients with respect to various chambers (in the sense of Dolgachev-Hu) form a family of Mori dream spaces, and they determine the pseudoeffective-, the movable-, and the nef cones in the Picard group of any of these quotients.

Britta Späth reformulates the inductive conditions for the conjectures by Alperin and McKay using (new) order relations between ordinary, respectively modular character triples. This allows to clarify the similarities and differences between those conditions.

Ulrich Thiel gives an overview of the theory of restricted rational Cherednik algebras. Their representation theory is connected to the geometry of the Calogero–Moser space, and there is a lot of evidence that they contain certain information about Hecke algebras. He outlines some open problems and conjectures, and determines explicitly the representation theory of restricted rational Cherednik algebras for dihedral groups.

Christoph Zellner shows in his article that under certain conditions, concerning in particular the structure of the Lie algebra $\mathfrak{g}$ of G, a continuous unitary representation of G is automatically smooth. As an application, this yields a dense space of smooth vectors for continuous positive energy representations of oscillator groups, double extensions of loop groups and the Virasoro group.

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Symmetric superspaces: slices, radial parts, and invariant functions

Alexander Alldridge*

Abstract. We study the restriction of invariant polynomials on the tangent space of a Riemannian symmetric supermanifold to a Cartan subspace. We survey known results in the case the symmetric space is a Lie supergroup, and more generally, where the Cartan subspace is even. We then describe an approach to this problem, developed in joint work in progress with K. Coulembier, based on the study of radial parts of differential operators. This leads to a characterisation of the invariant functions for an arbitrary linear isometric action, and as a special case, to a Chevalley restriction theorem valid for the isotropy representation of any contragredient Riemannian symmetric superspace.

2010 Mathematics Subject Classification. Primary 58C50, 58E40; Secondary 17B20, 17B35, 53C35.

Keywords. Chevalley restriction theorem, differential operator, Harish–Chandra homomorphism, Lie superalgebra, radial part, Riemannian symmetric superspace.

1. Introduction

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ a choice of Borel subalgebra containing $\mathfrak{h}$. Any regular semi-simple element of $\mathfrak{g}$ is upon the action of the adjoint group G of $\mathfrak{g}$ conjugate to an element of $\mathfrak{h}$, and the action of G is reflected on $\mathfrak{h}$ by the action of the Weyl group

$$W = W(\mathfrak{g} : \mathfrak{h}) := N_G(\mathfrak{h})/Z_G(\mathfrak{h}) = \big\langle s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta\rangle \cdot \alpha \;\big|\; \alpha \in \Delta(\mathfrak{g} : \mathfrak{h})\big\rangle. \quad (1.1)$$

One may ask to which extent the invariants in the symmetric and/or universal enveloping algebra of $\mathfrak{g}$ may be expressed in terms of data on the Cartan subalgebra. Classical theorems of C. Chevalley and Harish–Chandra [9] furnish the following answer to this question.

Theorem 1.1 (Chevalley, Harish–Chandra [9]).

(1) *Splitting* $\mathfrak{g}$ *as* $\mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{g}]$ *yields a restriction map* $\mathrm{res}_{\mathfrak{h}} : S(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$. *It induces an algebra isomorphism*

$$S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$$

whose image is a polynomial algebra in $r = \operatorname{rank} \mathfrak{g} = \dim \mathfrak{h}$ *indeterminates.*

*Research funded by Deutsche Forschungsgemeinschaft (DFG), grant nos. SFB/TR 12, ZI 513/2-1, and the Institutional Strategy of the University of Cologne in the Excellence Initiative.

(2) *There is an algebra isomorphism* $\Gamma : \mathcal{Z}(\mathfrak{g}) := \mathfrak{U}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$, *the* Harish–Chandra homomorphism, *given by the projection*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(h) \oplus (\mathfrak{u}\,\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\,\mathfrak{u}^-) \longrightarrow \mathfrak{U}(\mathfrak{h}) = S(\mathfrak{h}),$$

followed by the shift $p \longmapsto p(\cdot - \rho)$, $\rho := \mathrm{tr}_{\mathfrak{u}}\,\mathrm{ad}\,|_{\mathfrak{h}} = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

Of course, the first part of the theorem may be rephrased in terms of categorical quotients, but we will stick to the more algebraic perspective in this survey.

More generally, one may assume that θ is a Cartan involution of $\mathfrak{g}$, with eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and consider the action of the adjoint group K of $\mathfrak{k}$ in $\mathfrak{p}$. The previous situation is then recovered from the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ with the flip involution. This is commonly referred to as the "group case", since the corresponding symmetric space G/K is in this case a group.

In the general case, the Cartan subalgebra $\mathfrak{h}$ is then replaced by the choice of a Cartan subspace $\mathfrak{a} \subseteq \mathfrak{p}$, i.e. a maximal subalgebra consisting of semi-simple elements. A choice of positive roots Σ^+ in $\Sigma = \Delta(\mathfrak{g} : \mathfrak{a})$ then determines a minimal θ-parabolic $\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$ where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{n}$ is the sum of the positive $\mathfrak{a}$-root spaces.

With the Weyl group

$$W = W(\mathfrak{g} : \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a}) = \big\langle s_\lambda(\mu) = \mu - \langle\lambda^\vee, \mu\rangle \cdot \lambda \,\big|\, \lambda \in \Sigma\big\rangle,$$

we have the corresponding generalisation of Theorem 1.1.

Theorem 1.2 (Chevalley, Harish–Chandra [9]).

(1) *Splitting* $\mathfrak{p}$ *as* $\mathfrak{a} \oplus [\mathfrak{k}, \mathfrak{a}]$ *yields a restriction map* $\mathrm{res}_{\mathfrak{a}} : S(\mathfrak{p}) \longrightarrow S(\mathfrak{a})$. *It induces an algebra isomorphism*

$$S(\mathfrak{p})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$$

whose image is a polynomial algebra in $r = \mathrm{rank}\,\Sigma$ *indeterminates.*

(2) *There is an exact sequence*

$$0 \longrightarrow (\mathfrak{k}\,\mathfrak{U}(\mathfrak{k}))^{\mathfrak{k}} \longrightarrow \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \xrightarrow{\ \Gamma\ } S(\mathfrak{a})^W \longrightarrow 0$$

of algebras, where Γ *is given by the projection*

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{a}) \oplus (\mathfrak{n}\,\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\,\theta(\mathfrak{n})) \longrightarrow \mathfrak{U}(\mathfrak{a}) = S(\mathfrak{a}),$$

followed by the shift $p \longmapsto p(\cdot - \rho)$, $\rho := \mathrm{tr}_{\mathfrak{n}}\,\mathrm{ad}\,|_{\mathfrak{a}} = \frac{1}{2}\sum_{\lambda\in\Sigma^+}\dim\mathfrak{g}^\lambda \cdot \lambda$.

In this survey, we will consider the setting where $(\mathfrak{g}, \theta)$ is replaced by a reductive symmetric superpair where $\mathfrak{g}$ is a (finite-dimensional) contragredient Lie superalgebra. We will report on the state of the literature and on some recent progress achieved together with K. Coulembier (of the University of Sydney).

2. Graded group cases: Results of Sergeev, Kac, Gorelik

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a contragredient (finite-dimensional) Lie superalgebra [11, 14] and $\mathfrak{h} \subseteq \mathfrak{g}_{\bar{0}}$ a Cartan subalgebra. Let $\mathfrak{b} \subseteq \mathfrak{g}$ be a choice of Borel subalgebra containing $\mathfrak{h}$ and $W = W(\mathfrak{g}_{\bar{0}} : \mathfrak{h})$ the Weyl group of $\mathfrak{g}$.

It is natural to ask the same question about invariants that we considered above in this more general setting. However, the introduction of the $\mathbb{Z}/2\mathbb{Z}$ grading changes the situation drastically. Theorems of A. Sergeev [17], V. Kac [12], and M. Gorelik [8] in this regard may be summarised as follows.

Theorem 2.1 (Sergeev, Kac, Gorelik [8, 12, 17]).

(1) *The restriction map* $\operatorname{res}_{\mathfrak{h}}$ *induces an injection* $S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow S(\mathfrak{h})^W$ *whose image* $I(\mathfrak{h})$ *consists of all* $p \in S(\mathfrak{h})^W$ *such that*

$$p(\lambda + \alpha) = p(\lambda) \quad \forall \lambda \in \mathfrak{h}^*, \alpha \in \overline{\Delta}_{\bar{1}}, \langle \lambda + \rho, \alpha \rangle = 0.$$

Here, $\overline{\Delta}_{\bar{1}} := \Delta_{\bar{1}} \setminus \mathbb{Q}\Delta_{\bar{0}}$ *denotes the set of purely odd roots.*

(2) *The Harish–Chandra homomorphism* Γ *induces an injective algebra morphism* $\mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$ *whose image is* $I(\mathfrak{h})$.

Remark 2.2 (Sergeev, Stembridge [17, 18]). The algebra $I(\mathfrak{gl}(m|n))$ of invariants is not finitely generated if $m, n \geqslant 1$.

Thus, the mere introduction of the grading has destroyed finite generation and introduced apparently alien differential conditions into the picture. The origin of these will become apparent at the end of this survey.

Before we state our results in the case of symmetric Lie superalgebras, let us review their classification.

3. Symmetric contragredient Lie superalgebras

Let θ be an involution of $\mathfrak{g}$. V. Serganova [16] has classified conjugacy classes of involutions in all classical and some exceptional cases. Previously, M. Parker [15] had classified real forms of the simple contragredient Lie superalgebras, and these correspond bijectively to classes of involutions. Recently, Chuah [6] has given a classification of involutions of simple contragredient Lie superalgebras in terms of Vogan superdiagrams.

We are interested in the case where $\mathfrak{k} = \ker(1 - \theta)$ is a non-degenerate subspace for the invariant form on $\mathfrak{g}$. We call such symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ *Riemannian.*

Theorem 3.1 (Parker, Serganova [15, 16]). *Up to parity, conjugacy under the adjoint group of* $\mathfrak{g}_{\bar{0}}$ *in* $\mathfrak{g}$*, and the formation of simple subquotients and direct sums, the contragredient Riemannian symmetric superpairs* $(\mathfrak{g}, \mathfrak{k})$ *are one of the following:*

(1) *simple Lie algebra symmetric pairs,*

(2) *parity involution pairs* $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$, $\theta(x) = (-1)^{|x|}x$,

(3) *group type pairs* $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$, $\theta(x, y) = (y, x)$, *and*

(4) *entries in the following list:*

label	$\mathfrak{g}$	$\mathfrak{k}$
$A\mathrm{I}\|A\mathrm{II}$	$\mathfrak{gl}(p\|2q)$	$\mathfrak{osp}(p\|2q)$
$A\mathrm{III}\|A\mathrm{III}$	$\mathfrak{gl}(p+q\|r+s)$	$\mathfrak{gl}(p\|r) \times \mathfrak{gl}(q\|s)$
$BD\mathrm{I}\|C\mathrm{II}$	$\mathfrak{osp}(p+q\|2r+2s)$	$\mathfrak{osp}(p\|2r) \times \mathfrak{osp}(q\|2s)$
$D\mathrm{III}\|C\mathrm{I}$	$\mathfrak{osp}(2p\|2q)$	$\mathfrak{gl}(p\|q)$
$(BD\mathrm{I}\|C\mathrm{II})_\alpha$	$D(2,1;\alpha)$	$\mathfrak{osp}(2\|2) \times \mathfrak{o}(2)$
$BD\mathrm{I}_{\mathrm{I}}$	$F(4)$	$\mathfrak{sl}(4\|1)$
$BD\mathrm{I}_{\mathrm{II}}$	$F(4)$	$\mathfrak{osp}(2\|4) \times \mathbb{C}$
$BD\mathrm{I}_{\mathrm{III}}$	$F(4)$	$\mathfrak{osp}(4\|2) \times \mathfrak{sl}(2)$
G_{I}	$G(3)$	$\mathfrak{osp}(3\|2) \times \mathfrak{sl}(2)$
G_{II}	$G(3)$	$\mathfrak{osp}(4\|2)$

The main distinction of general contragredient symmetric superpairs (as compared to those of group type considered by Sergeev, Kac, and Gorelik) is that the choice of "Cartan subspace" is more delicate and may involve odd as well as even coordinate directions.

Definition 3.2. Let $(\mathfrak{g}, \theta)$ be a Riemannian symmetric contragredient Lie superalgebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}_{\bar{0}} \subseteq \mathfrak{p}_{\bar{0}}$ be a Cartan subspace for $(\mathfrak{g}_{\bar{0}}, \theta|_{\mathfrak{g}_{\bar{0}}})$ and define

$$\mathfrak{a} := \mathfrak{z}_{\mathfrak{p}}(\mathfrak{a}_{\bar{0}}).$$

Then $(\mathfrak{g}, \theta)$ (or $(\mathfrak{g}, \mathfrak{k})$) is called of *even type* if $\mathfrak{a} = \mathfrak{a}_{\bar{0}}$, and otherwise, of *odd type*. This definition does not depend on the choice of $\mathfrak{a}_{\bar{0}}$.

From the classification, the following is easy to read off.

Proposition 3.3. *Of the* $(\mathfrak{g}, \mathfrak{k})$ *listed above, the following are of even type:*

(1) *Lie algebra symmetric pairs,*

(2) *group type symmetric pairs,*

(3) $A\mathrm{I}|A\mathrm{II}$, $D\mathrm{III}|C\mathrm{I}$, $(BD\mathrm{I}|C\mathrm{II})_\alpha$, $BD\mathrm{I}_{\mathrm{I}}$, $BD\mathrm{I}_{\mathrm{II}}$, $BD\mathrm{I}_{\mathrm{III}}$, G_{I}, *and*

(4) *the types* $A\mathrm{III}|A\mathrm{III}$ *and* $BD\mathrm{I}|C\mathrm{II}$ *for* $(p-q)(r-s) \geqslant 0$.

The following are the odd type pairs:

(5) *parity involution pairs,*

(6) *type* G_{II}, *and*

(7) *the types* $A\mathrm{III}|A\mathrm{III}$ *and* $BD\mathrm{I}|C\mathrm{II}$ *for* $(p-q)(r-s) < 0$.

4. Results in even type

The first results for symmetric superpairs were obtained under the assumption of even type. As we shall explain below, with hindsight, it is clear that this situation is much more tractable than the general case.

The following is the main result obtained in joint work with J. Hilgert and M.R. Zirnbauer [5].

Theorem 4.1 (A.–Hilgert–Zirnbauer [5]). *The restriction map* $\mathrm{res}_{\mathfrak{a}}$ *induces an injective algebra morphism* $S(\mathfrak{p})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$ *whose image is*

$$I(\mathfrak{a}) := \bigcap_{\lambda \in \overline{\Sigma}_{\bar{1}}} I_\lambda,$$

where $\overline{\Sigma}_{\bar{1}} := \Sigma_{\bar{1}} \setminus \mathbb{Q}\Sigma_{\bar{0}}$ *denotes the set of purely odd restricted roots, for* $\langle \lambda, \lambda \rangle = 0$*:*

$$I_\lambda := \big\{ p \in S(\mathfrak{a})^W \mid \partial_\lambda^k p \in (\lambda^{\vee k}), k = 0, \ldots, \tfrac{1}{2} \dim \mathfrak{g}_{\bar{1}}^\lambda \big\},$$

and for $\langle \lambda, \lambda \rangle \neq 0$*:*

$$I_\lambda := \big\{ p \in S(\mathfrak{a})^W \mid \partial_\lambda^k p \in (\lambda^{\vee}), k = 1, 3, 5, \ldots, \dim \mathfrak{g}_{\bar{1}}^\lambda - 1 \big\}.$$

Remark 4.2. A striking new feature is the appearance of rational singularities in some cases where the algebra of invariants is finitely generated.

Consider e.g. $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{osp}(2|2q), \mathfrak{osp}(1|2q))$. Then

$$I(\mathfrak{a}) = \mathbb{C}[a^2, a^{2q+1}] \cong \mathbb{C}[X, Y]/(X^{2q+1} - Y^2).$$

Another new phenomenon is that the invariants in the universal enveloping algebra are no longer isomorphic to the algebra of invariant functions. Indeed, the following is the main result of [1].

Theorem 4.3 (A. [1]). *The Harish–Chandra homomorphism* $\Gamma : \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} \longrightarrow S(\mathfrak{a})^W$ *has kernel* $(\mathfrak{k}\,\mathfrak{U}(\mathfrak{g}))^{\mathfrak{k}}$ *and image*

$$J(\mathfrak{a}) := \bigcap_{\lambda \in \overline{\Sigma}_{\bar{1}}} J_\lambda$$

where for $\langle \lambda, \lambda \rangle = 0$*:*

$$J_\lambda := I_\lambda$$

and for $\langle \lambda, \lambda \rangle \neq 0$ *and* $2q = \dim \mathfrak{g}_{\bar{1}}^\lambda$*:*

$$J_\lambda := S(\mathfrak{a})^W \cap \mathbb{C}[(\check{\lambda}^2 - q^2), (\check{\lambda} - q)(\check{\lambda}^2 - q^2)^q].$$

We have $\mathrm{gr}\, J(\mathfrak{a}) = I(\mathfrak{a})$*, but in general* $J(\mathfrak{a}) \not\cong I(\mathfrak{a})$*.*

The proof of Theorem 4.1 is based on the isomorphism

$$\mathbb{C}[\mathfrak{p}^*] = \mathrm{Hom}_{S(\mathfrak{p}_{\bar 0})}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_{\bar 0}^*]),$$

so that a polynomial may be extended to a superpolynomial by defining the action of constant coefficient differential operators on it. Moreover, the action of differential operators on invariant polynomials is determined fully by their *radial part*, defined by the following proposition.

To that end, let $\mathfrak{a}_{\bar 0}' \subseteq \mathfrak{a}_{\bar 0}$ be the set of all *super-regular* elements $h \in \mathfrak{a}_{\bar 0}$, that is, $\lambda(h) \neq 0$ for all $\lambda \in \Sigma$. Moreover, let $\mathfrak{a}'$ be the open subspace of the locally super-ringed space attached to $\mathfrak{a}$.

Proposition 4.4. *For any constant coefficient differential operator D on $\mathfrak{p}$, there is a unique differential operator $\overline{D}$ on $\mathfrak{a}'$, the* radial part *of D, such that*

$$\overline{D}(f|_{\mathfrak{a}'}) = D(f)|_{\mathfrak{a}'}$$

for any locally defined $\mathfrak{k}$-invariant analytic function f.

This was established in Ref. [5] by purely algebraic means. Although this was sufficient for our purposes, the definition and the restriction to constant coefficient operators was somewhat unsatisfactory. A more general result, based on an approach more firmly rooted in supergeometry, will be presented below.

Now the basic observation in Ref. [5] was as follows.

Theorem 4.5 (A.–Hilgert–Zirnbauer [5]). *We have*

$$I_\lambda = \bigcap_{D \in S(\mathfrak{p}_{\bar 1}^\lambda)} \mathrm{dom}\,\overline{D}$$

where $\mathrm{dom}\,\overline{D}$ *is the domain of the differential operator $\overline{D}$ and $\mathfrak{p}_{\bar 1}^\lambda$ is the $\mathfrak{p}$-projection of the restricted root space $\mathfrak{g}_{\bar 1}^\lambda$.*

This was established by computing, for a symplectic basis $(z_i, \tilde z_i)$ of $\mathfrak{p}_{\bar 1}^\alpha$, the radials parts of the corresponding basis $z_I \tilde z_I$ of $S(\mathfrak{p}_{\bar 1}^\lambda)$ explicitly as

$$\overline{z_I \tilde z_I} = (-1)^{\frac{k(k+1)}{2}} \sum_{j=0}^{k-1} \frac{(k-1+j)!}{2^j (k-1-j)!} \frac{(-\langle \lambda, \lambda \rangle)^j}{\check\lambda^{k+j}} \partial_\lambda^{k-j}, \quad k = |I|. \qquad (4.1)$$

Although ultimately successful, this "brute force" approach did not shed too much light on the problem. The general impression was that a smaller set of differential operators (with possibly simple radial parts) should suffice to detect the possible singularities that a Weyl-group invariant polynomial might pick up upon extension to $\mathfrak{p}^*$.

Ideally, such a characterisation should "explain" the differential conditions appearing in the known characterisations of invariant polynomials, and also help in cases of odd type. In work in progress with K. Coulembier [2,3], we have been able to make sense of the notion of radial parts in general and have applied these results to the problem of determining invariant functions for isometric linear actions of supergroups, with particular attention to the case of symmetric superpairs. The remainder of this survey will be devoted to the description of these results.