



国外电气信息类优秀教材改编系列

Adapted Oversea Excellent Textbooks on Electrical Engineering

PEARSON

Prentice  
Hall

# 统计与随机过程 在信号处理中的应用

Probability and Random Processes with  
Applications to Signal Processing, 3/e

原著 Henry Stark John W. Woods

改编 孟桥



高等教育出版社  
Higher Education Press



国外电气信息类优秀教材改编系列

Adapted Oversea Excellent Textbooks on Electrical Engineering

PEARSON

Prentice  
Hall

# 统计与随机过程 在信号处理中的应用

Probability and Random Processes with  
Applications to Signal Processing, 3/e

原著 Henry Stark

John W. Woods

改编 孟桥

江苏工业学院图书馆  
藏书章

010-28281118 销售热线  
800-810-0298 商务电话  
http://www.hep.edu.cn 网址  
http://www.hep.com.cn 网上订购  
http://www.jandisco.com 网上订购  
http://www.jandisco.com.cn 网上订购  
http://www.widened.com 网上订购

2008年1月第1次印刷  
2008年1月第1次印刷  
72.00元 定价

010-28281000 销售热线  
100011 商务电话  
010-28281000 网上订购  
010-28281000 网上订购  
010-28281000 网上订购  
010-28281000 网上订购



高等教育出版社  
Higher Education Press

010-28281000 销售热线

Original edition, entitled PROBABILITY AND RANDOM PROCESS WITH APPLICATIONS TO SIGNAL PROCESSING, 3rd Edition by STARK, HENRY; WOODS, JOHN W. published by Pearson Education, Inc, publishing as Prentice Hall, Copyright © 2002.

All rights reserved. No part of this book may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage retrieval system, without permission from Pearson Education, Inc.

China Adapted edition published by Pearson Education Asia Ltd. And Higher Education Press, Copyright © 2008.

This Adapted edition is manufactured in the People's Republic of China, and is authorized for sale only in People's Republic of China excluding Hong Kong and Macau.

本书封面贴有 Pearson Education (培生教育出版集团) 激光防伪标签。无标签者不得销售。

For sale and distribution in the People's Republic of China exclusively (except Taiwan, Hong Kong SAR and Macao SAR).

仅限于中华人民共和国境内(但不允许在中国香港、澳门特别行政区和中国台湾地区)销售发行。

### 图书在版编目(CIP)数据

统计与随机过程在信号处理中的应用 = Probability and Random Processes with Applications to Signal Processing: 英文 / (美)斯塔克(Stark, H.), (美)伍兹(Woods, J. W.)著; 孟桥改编. —北京: 高等教育出版社, 2008.1

ISBN 978-7-04-022582-2

I. 统… II. ①斯…②伍…③孟… III. ①统计信号—信号处理—高等学校—教材—英文②随机信号—信号处理—高等学校—教材—英文 IV. TN911.7

中国版本图书馆 CIP 数据核字(2007)第 186797 号

出版发行	高等教育出版社	购书热线	010-58581118
社 址	北京市西城区德外大街 4 号	免费咨询	800-810-0598
邮政编码	100011	网 址	<a href="http://www.hep.edu.cn">http://www.hep.edu.cn</a>
总 机	010-58581000		<a href="http://www.hep.com.cn">http://www.hep.com.cn</a>
经 销	蓝色畅想图书发行有限公司	网上订购	<a href="http://www.landaco.com">http://www.landaco.com</a>
印 刷	涿州市星河印刷有限公司		<a href="http://www.landaco.com.cn">http://www.landaco.com.cn</a>
		畅想教育	<a href="http://www.widedu.com">http://www.widedu.com</a>
开 本	787×960 1/16	版 次	2008 年 1 月第 1 版
印 张	46	印 次	2008 年 1 月第 1 次印刷
字 数	870 000	定 价	57.00 元

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换。

版权所有 侵权必究

物料号 22582-00

---

# Preface

---

---

---

The first edition of this book (1986) grew out of a set of notes used by the authors to teach two one-semester courses on probability and random processes at Rensselaer Polytechnic Institute (RPI). At that time the probability course at RPI was required of all students in the Computer and Systems Engineering Program and was a highly recommended elective for students in closely related areas. While many undergraduate students took the course in the junior year, many seniors and first-year graduate students took the course for credit as well. Then, as now, most of the students were engineering students. To serve these students well, we felt that we should be rigorous in introducing fundamental principles while furnishing many opportunities for students to develop their skills at solving problems.

There are many books in this area and they range widely in their coverage and depth. At one extreme are the very rigorous and authoritative books that view probability from the point of view of measure theory and relate probability to rather exotic theorems such as the Radon-Nikodym theorem (see for example *Probability and Measure* by Patrick Billingsley, Wiley, 1978). At the other extreme are books that usually combine probability and statistics and largely omit underlying theory and the more advanced types of applications of probability. In the middle are the large number of books that combine probability and random processes, largely avoiding a measure theoretic approach, preferring to emphasize the axioms upon which the theory is based. It would be fair to say that our book falls into this latter category. Nevertheless this begs the question: why write or revise another book in this area if there are already several good texts out there that use the same approach and provide roughly the same coverage? Of course back in 1986 there were few books that emphasized the engineering applications of probability and random pro-

cesses and that integrated the latter into one volume. Now there are several such books.

Both authors have been associated (both as students and faculty) with colleges and universities that have demanding programs in engineering and applied science. Thus their experience and exposure have been to superior students that would not be content with a text that furnished a shallow discussion of probability. At the same time, however, the authors wanted to write a book on probability and random processes for engineering and applied science students. A measure-theoretic book, or one that avoided the engineering applications of probability and the processing of random signals, was regarded not suitable for such students. At the same time the authors felt that the book should have enough depth so that students taking 2<sup>nd</sup> year graduate courses in advanced topics such as estimation and detection, pattern recognition, voice and image processing, networking and queuing, and so forth would not be handicapped by insufficient knowledge of the fundamentals and applications of random phenomena. In a nutshell, we tried to write a book that combined rigor with accessibility and had a strong self-teaching orientation. To that end we included a large number of worked-out examples, MATLAB codes, and special appendices that include a review of the kind of basic math needed for solving problems in probability as well as an introduction to measure theory and its relation to probability. The MATLAB codes, as well as other useful material such as multiple choice exams that cover each of the book's sections, can be found at the book's web site <http://www.prenhall.com/stark>.

The normal use of this book would be as follows: for a first course in probability at, say the junior or senior year, a reasonable goal is to cover Chapters 1 through 4. Nevertheless we have found that this may be too much for students not well prepared in mathematics. In that case we suggest a load reduction in which *combinatorics* in Chapter 1 (parts of Section 1.8), *failure rates* in Chapter 2 (Section 2.7), *more advanced density functions* and the *Poisson transform* in Chapter 3 are lightly or not covered the first time around. The proof of the *Central Limit Theorem*, *joint characteristic functions*, and Section 4.8 which deals with statistics, all in Chapter 4, can, likewise, also be omitted on a first reading.

Chapters 5 to 9 provide the material for a first course in random processes. Normally such a course is taken in the first year of graduate studies and is required for all further study in signal processing, communications, computer and communication networking, controls, and estimation and detection theory. Here what to cov-

er is given greater latitude. If pressed for time, we suggest that the pattern recognition applications and simultaneous diagonalization of two covariance matrices in Chapter 5 be given lower preference than the other material in that chapter. Chapters 6 and 7 are essential for any course in random processes and the coverage of the topics therein should be given high priority. Chapter 9 on signal processing should likewise be given high priority, because it illustrates the applications of the theory to current state-of-art problems. However, within Chapter 9, the instructor can choose among a number of applications and need not cover them all if time pressure becomes an issue, Chapter 8 dealing with advanced topics is critically important to the more advanced students, especially those seeking further studies toward the Ph. D. Nevertheless it too can be lightly covered or omitted in a first course if time is the critical factor.

Readers familiar with the 2<sup>nd</sup> edition of this book will find significant changes in the 3<sup>rd</sup> edition. The changes were the result of numerous suggestions made by lecturers and students alike. To begin with, we modified the title to *Probability and Random Processes with Applications to Signal Processing*, to better reflect the contents. We removed the two chapters on estimation theory and moved some of this material to other chapters where it naturally fitted in with the material already there. Some of the material on parameter estimation e. g. , the Gauss-Markov Theorem has been removed, owing to the need for finding space for new material. In terms of organization, the major changes have been in the random processes part of the book. Many readers preferred seeing discrete-time random phenomena in one chapter and continuous-time phenomena in another chapter. In the earlier editions of the book there was a division along these lines but also a secondary division along the lines of stationary versus non-stationary processes. For some this made the book awkward to teach from. Now all of the material on discrete-time phenomena appears in one chapter (Chapter 6); likewise for continuous-time phenomena (Chapter 7). Another major change is a new Chapter 9 that discusses applications to signal processing. Included are such topics as: the orthogonality principle, Wiener and Kalman filters, The Expectation-Maximization algorithm, Hidden Markov Models, and simulated annealing. Chapter 8 (Advanced Topics) covers much of the same ground as the old Chapter 9 e. g. , stochastic continuity, meansquare convergence, Ergodicity etc. and material from the old Chapter 10 on representation of random processes.

There have been significant changes in the first half of the book also. For ex-

ample, in Chapter 1 there is an added section on the misuses of probability in ordinary life. Here we were helped by the discussions in Steve Pinker's excellent book *How the Mind Works* (Norton Publishers, New York, 1997). Chapter 2 (*Random Variables*) now includes discussions on more advanced distributions such as the Gamma, Chi-square and the Student-t. All of the chapters have many more worked-out examples as well as more homework problems. Whenever convenient we tried to use MATLAB to obtain graphical results. Also, being a book primarily for engineers, many of the worked-out example and homework problems relate to real-life systems or situations.

We have added several new appendices to provide the necessary background mathematics for certain results in the text and to enrich the reader's understanding of probability. An appendix on Measure Theory falls in the latter category. Among the former are appendices on the delta and gamma functions, probability-related basic math, including the principle of proof-by-induction, Jacobians for  $n$ -dimensional transformations, and material on Fourier and Laplace inversion.

For this edition, the authors would like to thank Geoffrey Williamson and Yongyi Yang for numerous insightful discussions and help with some of the MATLAB programs. Also we thank Nikos Galatsanos, Miles Wernick, Geoffrey Chan, Joseph LoCicero, and Don Ucci for helpful suggestions. We also would like to thank the administrations of Illinois Institute of Technology and Rensselaer Polytechnic Institute for their patience and support while this third edition was being prepared. Of course, in the end, it is the reaction of the students that is the strongest driving force for improvements. To all our students and readers we owe a large debt of gratitude.

Henry Stark  
John W. Woods

---

# Contents

---

---

---

---

Preface	IX
1 Introduction to Probability	1
1.1 INTRODUCTION: WHY STUDY PROBABILITY?	1
1.2 THE DIFFERENT KINDS OF PROBABILITY	3
A. Probability as Intuition	3
B. Probability as the Ratio of FAVORABLE to Total Outcomes (Classical Theory)	3
C. Probability as a Measure of Frequency of Occurrence	5
D. Probability Based on an Axiomatic Theory	6
1.3 MISUSES, MISCALCULATIONS, AND PARADOXES IN PROBABILITY	6
1.4 SETS, FIELDS, AND EVENTS	8
Examples of Sample Spaces	9
1.5 AXIOMATIC DEFINITION OF PROBABILITY	13
1.6 JOINT, CONDITIONAL, AND TOTAL PROBABILITIES; INDEPENDENCE	18
1.7 BAYES' THEOREM AND APPLICATIONS	25
1.8 COMBINATORICS	27
Occupancy Problems	31
Extensions and Applications	34
1.9 BERNOULLI TRIALS—BINOMIAL AND MULTINOMIAL PROBABILITY LAWS	36
Multinomial Probability Law	40



1.10	ASYMPTOTIC BEHAVIOR OF THE BINOMIAL LAW: THE POISSON LAW	44
1.11	NORMAL APPROXIMATION TO THE BINOMIAL LAW	50
1.12	SUMMARY	52
	PROBLEMS	53
	REFERENCES	64
<b>2</b>	<b>Random Variables</b>	<b>66</b>
2.1	INTRODUCTION	66
2.2	DEFINITION OF A RANDOM VARIABLE	67
2.3	PROBABILITY DISTRIBUTION FUNCTION	71
2.4	PROBABILITY DENSITY FUNCTION(pdf)	75
	Four Other Common Density Functions	80
	More Advanced Density Functions	83
2.5	CONTINUOUS, DISCRETE, AND MIXED RANDOM VARIABLES	84
	Examples of Probability Mass Functions	86
2.6	CONDITIONAL AND JOINT DISTRIBUTIONS AND DENSITIES	90
2.7	FAILURE RATES	114
2.8	FUNCTIONS OF A RANDOM VARIABLE	118
2.9	SOLVING PROBLEMS OF THE TYPE $Y = g(X)$	123
2.10	SOLVING PROBLEMS OF THE TYPE $Z = g(X, Y)$	131
2.11	SOLVING PROBLEMS OF THE TYPE $V = g(X, Y)$ , $W = h(X, Y)$	149
2.12	SUMMARY	151
	PROBLEMS	151
	REFERENCES	163
	ADDITIONAL READING	164
<b>3</b>	<b>Expectation and Introduction to Estimation</b>	<b>165</b>
3.1	EXPECTED VALUE OF A RANDOM VARIABLE	165
	On the Validity of Equation 3.1-8	168
3.2	CONDITIONAL EXPECTATIONS	180
	Conditional Expectation as a Random Variable	187

---

3.3	MOMENTS	189
	Joint Moments	193
	Properties of Uncorrelated Random Variables	195
	Jointly Gaussian Random Variables	198
	Contours of Constant Density of the Joint Gaussian pdf	200
3.4	CHEBYSHEV AND SCHWARZ INEQUALITIES	203
	Random Variables with Nonnegative Values	205
	The Schwarz Inequality	206
3.5	MOMENT-GENERATING FUNCTIONS	208
3.6	CHERNOFF BOUND	211
3.7	CHARACTERISTIC FUNCTIONS	213
	Joint Characteristic Functions	219
	The Central Limit Theorem	221
3.8	ESTIMATORS FOR THE MEAN AND VARIANCE OF THE NORMAL LAW	226
	Confidence Intervals for the Mean	227
	Confidence Interval for the Variance	230
3.9	SUMMARY	234
	PROBLEMS	235
	REFERENCES	240
	ADDITIONAL READING	241
4	Random Vectors and Parameter Estimation	242
4.1	JOINT DISTRIBUTION AND DENSITIES	242
4.2	EXPECTATION VECTORS AND COVARIANCE MATRICES	247
4.3	PROPERTIES OF COVARIANCE MATRICES	249
4.4	SIMULTANEOUS DIAGONALIZATION OF TWO COVARIANCE MATRICES AND APPLICATIONS IN PATTERN RECOGNITION	254
	Projection	257
	Maximization of Quadratic Forms	258
4.5	THE MULTIDIMENSIONAL GAUSSIAN (NORMAL) LAW	264
4.6	CHARACTERISTIC FUNCTIONS OF RANDOM VECTORS	272
	The Characteristic Function of the Gaussian (Normal) Law	275
4.7	PARAMETER ESTIMATION	276

---

Estimation of $E[X]$	278
4.8 ESTIMATION OF VECTOR MEANS AND COVARIANCE MATRICES	281
Estimation of $\mu$	281
Estimation of the Covariance $K$	282
4.9 MAXIMUM LIKELIHOOD ESTIMATORS	284
4.10 LINEAR ESTIMATION OF VECTOR PARAMETERS	289
4.11 SUMMARY	292
PROBLEMS	294
REFERENCES	299
ADDITIONAL READING	299
 5 Random Sequences	 300
5.1 BASIC CONCEPTS	301
Infinite-Length Bernoulli Trials	307
Continuity of Probability Measure	312
Statistical Specification of a Random Sequence	314
5.2 BASIC PRINCIPLES OF DISCRETE-TIME LINEAR SYSTEMS	332
5.3 RANDOM SEQUENCES AND LINEAR SYSTEMS	339
5.4 WSS RANDOM SEQUENCES	347
Power Spectral Density	350
Interpretation of the PSD	352
Synthesis of Random Sequences and Discrete-Time Simulation	355
Decimation	358
Interpolation	359
5.5 MARKOV RANDOM SEQUENCES	362
ARMA Models	365
Markov Chains	366
5.6 VECTOR RANDOM SEQUENCES AND STATE EQUATIONS	374
5.7 CONVERGENCE OF RANDOM SEQUENCES	376
5.8 LAWS OF LARGE NUMBERS	385
5.9 SUMMARY	390
PROBLEMS	390
REFERENCES	403

---

<b>6</b>	<b>Random Processes</b>	<b>404</b>
6.1	BASIC DEFINITIONS	405
6.2	SOME IMPORTANT RANDOM PROCESSES	410
	Asynchronous Binary Signaling	410
	Poisson Counting Process	412
	Alternative Derivation of Poisson Process	416
	Random Telegraph Signal	419
	Digital Modulation Using Phase-Shift Keying	420
	Wiener Process or Brownian Motion	422
	Markov Random Processes	426
	Birth-Death Markov Chains	431
	Chapman-Kolmogorov Equations	435
	Random Process Generated from Random Sequences	436
6.3	CONTINUOUS-TIME LINEAR SYSTEMS WITH RANDOM INPUTS	436
	White Noise	441
6.4	SOME USEFUL CLASSIFICATIONS OF RANDOM PROCESSES	442
	Stationarity	443
6.5	WIDE-SENSE STATIONARY PROCESSES AND LSI SYSTEMS	445
	Wide-Sense Stationary Case	446
	Power Spectral Density	449
	An Interpretation of the psd	451
	More on White Noise	455
	Stationary Processes and Differential Equations	462
6.6	PERIODIC AND CYCLOSTATIONARY PROCESSES	465
6.7	VECTOR PROCESSES AND STATE EQUATIONS	471
	State Equations	474
6.8	SUMMARY	476
	PROBLEMS	477
	REFERENCES	495
<b>7</b>	<b>Advanced Topics in Random Processes</b>	<b>497</b>
7.1	MEAN-SQUARE(m.s.) CALCULUS	497

---

Stochastic Continuity and Derivatives[7-1]	497
Further Results on m. s. Convergence[7-1]	508
7.2 MEAN-SQUARE STOCHASTIC INTEGRALS	513
7.3 MEAN-SQUARE STOCHASTIC DIFFERENTIAL EQUATIONS	517
7.4 ERGODICITY[7-3]	522
7.5 KARHUNEN-LOEVE EXPANSION[7-5]	530
7.6 REPRESENTATION OF BANDLIMITED AND PERIODIC PROCESSES	536
Bandlimited Processes	537
Bandpass Random Processes	540
WSS Periodic Processes	543
Fourier Series for WSS Processes	546
7.7 SUMMARY	548
7.8 APPENDIX: INTEGRAL EQUATIONS	548
Existence Theorem	549
PROBLEMS	552
REFERENCES	565
 8 Applications to Statistical Signal Processing	 567
8.1 ESTIMATION OF RANDOM VARIABLES	567
More on the Conditional Mean	573
Orthogonality and Linear Estimation	575
Some Properties of the Operator $\hat{E}$	584
8.2 INNOVATION SEQUENCES AND KALMAN FILTERING	585
Predicting Gaussian Random Sequences	590
Kalman Predictor and Filter	591
Error-Covariance Equations	597
8.3 WIENER FILTERS FOR RANDOM SEQUENCES	601
Unrealizable Case(Smoothing)	602
Causal Wiener Filter	604
8.4 EXPECTATION-MAXIMIZATION ALGORITHM	605
Log-likelihood for the Linear Transformation	609
Summary of the E-M algorithm	611
E-M Algorithm for Exponential Probability Functions	611
Application to Emission Tomography	613

Log-likelihood Function of Complete Data	615
E-step	616
M-step	617
8.5 HIDDEN MARKOV MODELS(HMM)	617
Specification of an HMM	619
Application to Speech Processing	623
Efficient Computation of $P[E M]$ with a Recursive Algorithm	625
Viterbi Algorithm and the Most Likely State Sequence for the Observations	627
8.6 SPECTRAL ESTIMATION	630
The Periodogram	631
Bartlett's Procedure—Averaging Periodograms	634
Parametric Spectral Estimate	638
Maximum Entropy Spectral Density	640
8.7 SIMULATED ANNEALING	644
Gibbs Sampler	645
Noncausal Gauss-Markov Models	647
Compound Markov Models	651
Gibbs Line Sequence	652
8.8 SUMMARY	656
PROBLEMS	657
REFERENCES	661
Appendix A Review of Relevant Mathematics	664
A.1 BASIC MATHEMATICS	664
Sequences	664
Convergence	665
Summations	666
Z-Transform	666
A.2 CONTINUOUS MATHEMATICS	668
Definite and Indefinite Integrals	668
Differentiation of Integrals	668
Integration by Parts	669
Completing the Square	670
Double Integration	671

---

Functions	671
A.3 RESIDUE METHOD FOR INVERSE FOURIER TRANSFORMATION	673
Fact	674
Inverse Fourier Transform for psd of Random Sequence	677
A.4 MATHEMATICAL INDUCTION[A-4]	680
Axiom of Induction	680
REFERENCES	680
 Appendix B Gamma and Delta Functions	682
B.1 GAMMA FUNCTION	682
B.2 DIRAC DELTA FUNCTION	683
 Appendix C Functional Transformations and Jacobians	686
C.1 INTRODUCTION	686
C.2 JACOBIANS FOR $n = 2$	687
C.3 JACOBIAN FOR GENERAL $n$	689
 Appendix D Measure and Probability	691
D.1 INTRODUCTION AND BASIC IDEAS	691
Measurable Mappings and Functions	693
D.2 APPLICATION OF MEASURE THEORY TO PROBABILITY	694
Distribution Measure	694
 Appendix E Sampled Analog Waveforms and Discrete-time Signals	695
 Index	698

# 1

---

## ***Introduction to Probability***

---

---

---

### **1.1 INTRODUCTION: WHY STUDY PROBABILITY?**

One of the most frequent questions posed by beginning students of probability is : "Is anything truly random and if so how does one differentiate between the truly random and that which, because of a lack of information, is treated as random but really isn't?" First, regarding the question of truly random phenomena: "Do such things exist?" A theologian might state the case as follows: "We cannot claim to know the Creator's mind, and we cannot predict His actions because He operates on a scale too large to be perceived by man. Hence there are many things we shall never be able to predict no matter how refined our measurements."

At the other extreme from the cosmic scale is what happens at the atomic level. Our friends the physicists speak of such things as the *probability* of an atomic system being in a certain state. The uncertainty principle says that, try as we might, there is a limit to the accuracy with which the position and momentum can be simultaneously ascribed to a particle. Both quantities are fuzzy and indeterminate.

Many, including some of our most famous physicists, believe in an essential randomness of nature. Eugen Merzbacher in his well-known textbook on quantum mechanics[1-1] writes:



The probability doctrine of quantum mechanics asserts that the indetermination, of which we have just given an example, is a property inherent in nature and not merely a profession of our temporary ignorance from which we expect to be relieved by a future better and more complete theory. The conventional interpretation thus denies the possibility of an ideal theory which would encompass the present quantum mechanics but would be free of its supposed defects, the most notorious "imperfection" of quantum mechanics being the abandonment of strict classical determinism.

But the issue of determinism versus inherent indeterminism need never even be considered when discussing the validity of the probabilistic approach. The fact remains that there is, quite literally, a nearly uncountable number of situations where we cannot make any categorical deterministic assertion regarding a phenomenon because we cannot measure all the contributing elements. Take, for example, predicting the value of the current  $i(t)$  produced by a thermally excited resistor  $R$ . Conceivably, we might accurately predict  $i(t)$  at some instant  $t$  in the future if we could keep track, say, of the  $10^{23}$  or so excited electrons moving in each other's magnetic fields and setting up local field pulses that eventually all contribute to producing  $i(t)$ . Such a calculation is quite inconceivable, however, and therefore we use a probabilistic model rather than Maxwell's equations to deal with resistor noise. Similar arguments can be made for predicting weather, the outcome of a coin toss, the time to failure of a computer, and many other situations.

Thus to conclude: Regardless of which position one takes, that is, determinism versus indeterminism, we are forced to use probabilistic models in the real world because we do not know, cannot calculate, or cannot measure all the forces contributing to an effect. The forces may be too complicated, too numerous, or too faint.

Probability is a mathematical model to help us study physical systems in an *average sense*. Thus we cannot use probability in any meaningful sense to answer questions such as: "What is the probability that a comet will strike the earth tomorrow?" or "What is the probability that there is life on other planets?"<sup>①</sup>

R. A. Fisher and R. Von Mises, in the first third of the twentieth century, were largely responsible for developing the groundwork of modern probability theory. The modern axiomatic treatment upon which this book is based is largely the re-

---

① Nevertheless, certain evangelists and others have dealt with this question rather fearlessly. However, whatever probability system these people use, it is not the system that we shall discuss in this book.