

# Codes, Cryptology and Curves with Computer Algebra

Ruud Pellikaan, Xin-Wen Wu,  
Stanislav Bulygin and Relinde Jurrius



This well-balanced text touches on theoretical and applied aspects of protecting digital data. The reader is provided with the basic theory and is then shown deeper fascinating detail, including the current state of the art. Readers will soon become familiar with methods of protecting digital data while it is transmitted, as well as while the data is being stored.

Both basic and advanced error-correcting codes are introduced together with numerous results on their parameters and properties. The authors explain how to apply these codes to symmetric and public key cryptosystems and secret sharing. Interesting approaches based on polynomial systems solving are applied to cryptography and decoding codes. Computer algebra systems are also used to provide an understanding of how objects introduced in the book are constructed, and how their properties can be examined. This book is designed for Masters-level students studying mathematics, computer science, electrical engineering or physics.

**RUUD PELLIKAAN** has tenure at the Technische Universiteit Eindhoven, The Netherlands where his research has shifted from a devotion to coding theory, particularly algebraic geometry codes and their decoding, to code-based cryptography. He previously served as an associate editor of the *IEEE Transactions of Information Theory* and has organised several conferences.

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If three men be walking together,  
and (only) one of them be under a delusion,  
they may yet reach their goal, the deluded being the fewer;  
but if two of them be under the delusion, they will not do so,  
the deluded being the majority.

*“Heaven and Earth” chapter 14*  
*Zhuangzi (370–287 BC) [364]*





# Preface

An early version of this book was a handwritten manuscript from around 1990. In June 2001 a synopsis was written by invitation of Cambridge University Press with the working title “The construction and decoding of algebraic geometry codes,” or “Algebraic geometry and its applications (in error-correcting codes and cryptography).” That proposal was accepted, but with no indication of a deadline. So originally its aim was a book on algebraic geometry codes. As time passed more and more co-authors joined the team: Xin-Wen Wu in 2004, Stanislav Bulygin in 2007 and finally Relinde Jurrius in 2012.

Early versions of chapters were written on algebraic geometry codes, elementary coding theory, list decoding Reed–Muller codes, decoding algorithms and Gröbner bases, cryptography, weight enumerators and its generalizations and relations with matroid theory that appeared in books and journals [68, 69, 70, 71, 177, 184, 186, 266].

The prerequisites of this book are: elementary logical reasoning and naive set theory, some combinatorics and probability theory. Furthermore: linear algebra, the beginnings of group theory, the algebra of rings and fields. We will go into the details of polynomial rings and finite fields in several chapters.

The first six chapters of the book on the construction, properties and decoding of error-correcting codes are self-contained. This can be used as a course in Coding Theory of four hours a week during a semester in the first year of the Masters. It is advised to use Chapter 12 from the start to practise the theory with computer algebra systems.

The second half of the book on complexity theory, cryptology, Gröbner bases applied to codes and cryptosystems and algebraic geometry codes is more advanced. This can be used for a course in the second year of a Masters degree or can be read individually as a *Capita Selecta*. It is

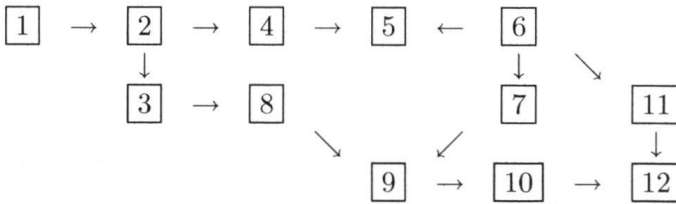
also a good starting point for a project or assignment. In the Notes at the end of every chapter we give ample references to further reading.

We thank the editors of Cambridge University Press: David Tranah, Jonathan Walthoe, Roger Astley, Clare Dennison and Abigail Walkington for their advice and patience.

The logical dependency between the chapters

- 1 Error-correcting codes
- 2 Code constructions and bounds
- 3 Weight enumeration
- 4 Cyclic codes
- 5 Polynomial codes
- 6 Algebraic decoding
- 7 Complexity and decoding
- 8 Codes and related structures
- 9 Cryptology
- 10 Gröbner bases for coding and cryptology
- 11 Codes on curves
- 12 Coding and cryptology with computer algebra

is given in the following diagram:



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# 1

## Error-correcting Codes

Ruud Pellikaan and Xin-Wen Wu

The idea of *redundant* information is a well-known phenomenon in reading a newspaper. Misspellings usually go unnoticed for a casual reader, while the meaning is still grasped. In Semitic languages such as Hebrew, and even older in the hieroglyphics in the tombs of the pharaohs of Egypt, only the consonants are written while the vowels are left out so that we do not know for sure how to pronounce these words nowadays. The letter “e” is the most frequent occurring symbol in the English language, and leaving out all these letters would still give in almost all cases an understandable text to the expense of greater attention of the reader.

The science of deleting redundant information in a clever way such that it can be stored in less memory or space and still can be expanded to the original message is called *data compression* or *source coding*. It is not the topic of this book. So we can compress data, but an error made in a compressed text would give a different message that is most of the time utterly meaningless.

The idea in *error-correcting codes* is the converse. One adds redundant information in such a way that it is possible to detect or even correct errors after transmission. In radio contacts between pilots and radar controls the letters in the alphabet are spoken phonetically as “Alpha, Bravo, Charlie, ...” but “Adams, Boston, Chicago, ...” is more commonly used for spelling in a telephone conversation. The addition of a parity check symbol enables one to detect an error, such as on the former punch cards that were fed into a computer, in the ISBN code for books, the European Article Numbering (EAN) and the Universal Product Code (UPC) for articles. Error-correcting codes are common



in numerous modern applications where data are required to be stored, processed and transmitted in a reliable manner, such as in audiovisual media, quick response (QR) codes, fault-tolerant computer systems and deep space telecommunication, to name but a few.

In this chapter, we present an introduction to error-correcting codes focusing on the most commonly used codes, namely, block codes and linear codes, including fundamental concepts and procedures for code construction, encoding and decoding.

## 1.1 Block Codes

Legend goes that Hamming was so frustrated the computer halted every time it detected an error after he handed in a stack of punch cards, he thought about a way the computer would be able not only to detect the error but also to correct it automatically. He came up with his nowadays famous code named after him. Whereas the theory of Hamming is about the actual construction, the encoding and decoding of codes and uses tools from *combinatorics* and *algebra*, the approach of Shannon leads to *information theory* and his theorems tell us what is and what is not possible in a *probabilistic* sense.

According to Shannon we have a message  $\mathbf{m}$  in a certain alphabet and of a certain length, we encode  $\mathbf{m}$  to  $\mathbf{c}$  by expanding the length of the message and adding redundant information. One can define the *information rate*  $R$  that measures the slowing down of the transmission of the data. The encoded message  $\mathbf{c}$  is sent over a noisy channel such that the symbols are changed, according to certain probabilities that are characteristic of the channel. The received word  $\mathbf{r}$  is decoded to  $\mathbf{m}'$ . See Figure 1.1. Now given the characteristics of the channel one can define the *capacity*  $C$  of the channel and it has the property that for every  $R < C$  it is possible to find an encoding and decoding scheme such that the *error probability* that  $\mathbf{m}' \neq \mathbf{m}$  is arbitrarily small. For  $R > C$  such

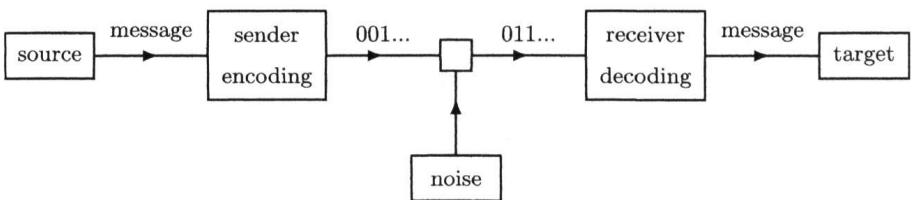


Figure 1.1 Block diagram of a communication system