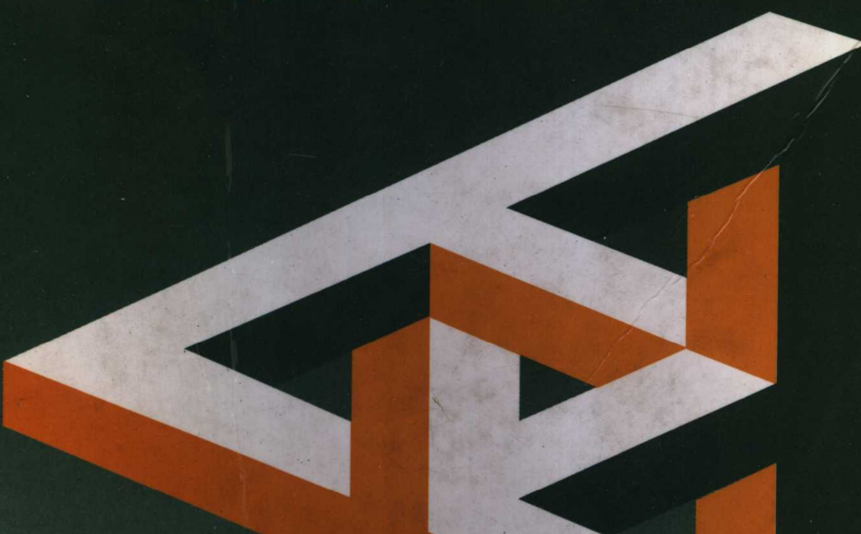
NONLINEAR FUNCTIONAL ANALYSIS



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PREFACE

This book presents the central ideas of applicable functional analysis in a vivid and straightforward fashion with a minimum of fuss and formality.

The book was developed while teaching an upper-division course in non-linear functional analysis. My intention was to give the background for the solution of nonlinear equations in Banach Spaces, and this is at least one intention of applicable functional analysis. This course is designed for a one-semester introduction at post-graduate level. However, the material can easily be expanded to fill a two semester-course.

To clarify what I taught, I wrote down each delivered lecture. The prerequisites for this text are basic theory on Analysis and Linear Functional Analysis. Any student with a certain amount of mathematical maturity will be able to read the book.

The material covered is more or less prerequisite for the students doing research in applicable mathematics. This text could thus be used for an M.Phil. course in the mathematics.

The preparation of this manuscript was possible due to the excellent facilities available at the Technomathematics Research Foundation, Kolhapur. I thank my colleagues and friends for their comments and help.

I specially thank Mrs. Achala Sabne for the excellent job of preparing the camera ready text.

Most of all, I would like to express my deepest gratitude to Rupali, my wife, in whose space and time this book was written.

R. AKERKAR

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Chapter 1

CONTRACTION

1.1 Banach's Fixed Point Theorem

Let $(X, d), (Y, d)$ be metric spaces. A mapping $F : X \to Y$ is said to be **Lipschitz continuous**, if there exists a constant $k > 0$, such that for all $x_1, x_2 \in X$

$$d(F(x_1), F(x_2)) \leq k.d(x_1, x_2).$$

F is called a **contraction**, if for all $x_1, x_2 \in X, x_1 \neq x_2$

$$d(F(x_1), F(x_2)) < d(x_1, x_2).$$

F is called a **strict or a k-contraction**, if F is Lipschitz continuous with a Lipschitz constant $k < 1$.

If $X \subset Y, F : X \to Y$, then $\hat{x} \in X$ is called a **fixed point** of F, if $F(\hat{x}) = \hat{x}$.

If an equation

$$H(x) = y \tag{1.1}$$

is to be solved, where $H : U \to X$ is a continuous mapping from a subset U of a normed space X into X, then this equation can be transformed in a fixed point problem :

Let $T : X \to X$ be an injective (linear) operator, then (1.1) is equivalent to

$$\begin{aligned} TH(x) &= Ty \\ x &= x - TH(x) + Ty \end{aligned}$$

hence

$$x = F(x) \tag{1.2}$$

where $F(x) = x - TH(x) + Ty$.

The (unique) fixed point x of (1.2) is a (the unique) solution of (1.1), since T is injective. T can be chosen, such that some fixed point principles are applicable. Now we will start with the most important fixed point theorem.

Theorem 1.1 *(Banach's Fixed Point Principle)*

Let X be a complete metric space. Let $F : X \to X$ be a k-contraction with $0 < k < 1$, i.e.

$$\forall \;\; x_1, x_2 \in X \;\; d(F(x_1), F(x_2)) \leq k.d(x_1, x_2).$$

Then the following hold

1^0 *There exists a fixed point $\hat{x}$ of F.*

2^0 *This fixed point is unique.*

3^0 *If $x_0 \in X$ is arbitrarily chosen, then the sequence (x_n), defined by $x_n = F(x_{n-1})$ converges to $\hat{x}$.*

4^0 *For all n the error estimate is true*

$$d(x_n, \hat{x}) \leq \frac{k}{1-k} d(x_n, x_{n-1}) \leq \frac{k^n}{1-k} d(x_1, x_0).$$

Proof :

For $x_0 \in X$ we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n), F(x_{n-1})) \leq k.d(x_n, x_{n-1}) \\ &\leq ... \leq k^n.d(x_1, x_0) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+j+1}, x_n) &\leq d(x_{n+j+1}, x_{n+j}) + ... + d(x_{n+1}, x_n) \\ &\leq (k^{j+1} + ... + k).d(x_n, x_{n-1}) \\ &\leq \frac{k}{1-k}.d(x_n, x_{n-1}) \\ &\leq \frac{k^n}{1-k}.d(x_1, x_0). \end{aligned}$$

Since $k < 1$, the sequence (x_n) is a Cauchy sequence. Since X is complete, $\lim x_n = \hat{x}$ exists. By continuity of F, we have

$$F(\hat{x}) = \lim F(x_n) = \lim x_{n+1} = \hat{x},$$

hence $\hat{x}$ is a fixed point of F.

If $\tilde{x}$ is a fixed point of F, then

$$d(\tilde{x}, \hat{x}) = d(F(\tilde{x}), F(\hat{x})) \leq k.d(\tilde{x}, \hat{x})$$

implies $\tilde{x} = \hat{x}$ (k is less than 1!) and we obtain the error estimates by

$$\begin{aligned} \lim_{j\to\infty} d(x_{n+j-1}, x_n) &= d(\hat{x}, x_n) \leq \frac{k}{1-k}.d(x_n, x_{n-1}) \\ &\leq \frac{k^n}{1-k}.d(x_1, x_0). \end{aligned}$$

□

This theorem meets all requirements for a useful mathematical statement: Existence, Uniqueness, Construction and Error Estimate.

If not necessarily F itself, but almost all iterates

$$F^n = FoF^{n-1}$$

are k_n-contractions, we obtain the following result.

Theorem 1.2 *Let X be a complete metric space, for $F : X \to X$ we assume:*

There exists a sequence (k_n) of positive reals, such that for all $x, y \in X$

$$\begin{aligned} d(F^n x, F^n y) &\leq k_n.d(x,y) \\ \Sigma k_n &< \infty. \end{aligned}$$

Then F has a unique fixed point $\hat{x}$, and $\hat{x} = lim x_n = lim F^n(x_0)$ with

$$d(x_n, \hat{x}) \leq \sum_{j \geq n} k_j.d(x_1, x_0).$$

Proof :

This proof is analogous to the proof of Theorem 1.1.

$$\begin{aligned} d(x_{n+j+1}, x_n) &\leq d(x_{n+j+1}, x_{n+j}) + ... + d(x_{n+1}, x_n) \\ &\leq d(F^{n+j}x_1, F^{n+j}x_0) + ... + d(F^n x_1, F^n x_0) \\ &\leq (k_{n+j} + ... + k_n).d(x_1, x_0). \end{aligned}$$

Thus, (x_n) is a Cauchy sequence. Let $\hat{x} = \lim x_n$, then $F(\hat{x}) = \lim F(x_n) = \lim x_{n+1} = \hat{x}$, i.e. $\hat{x}$ is fixed point; if $\tilde{x}$ is a fixed point of F, so $\tilde{x}$ is a fixed point for all F^n, hence

$$d(F^n \hat{x}, F^n \tilde{x}) \leq k_n.d(\hat{x}, \tilde{x}),$$

implies $\tilde{x} = \hat{x}$, since $k_n < 1$ for almost all n and

$$d(\hat{x}, x_n) \leq \lim_{j \to \infty} d(x_{n+j+1}, x_n) \leq \sum_{j \geq n} k_j.d(x_1, x_0). \qquad \square$$

If F is just a contraction, then F does not necessarily have a fixed point:

Let $X = [0, \infty)$ and $F : X \to X$ be defined by

$$F(x) = x + \frac{1}{x+1}.$$

$F(x) = x + \frac{1}{x+1} \neq x$, but

$$F(x) - F(y) = F'(\xi)(x-y) = \left[1 - \frac{1}{(1+\xi)^2}\right](x-y)$$

i.e. $1 - \frac{1}{(1+\xi)^2} < 1$, thus, if $x \neq y$,

$$|F(x) - F(y)| < |x - y|.$$

If we additionally assume that (X, d) is a compact metric space, then we obtain the following result.

Theorem 1.3 *Let X be a compact metric space, $F : X \to X$ a contraction. Then F has a unique fixed point and $\hat{x}$ with* $\hat{x} = lim x_n, \quad x_n = F(x_{n-1}), x_0 \in X$.

Proof :

Since X is compact, the sequence $(F(x_n))$ has a convergent subsequence $(F(x_{n_j}))$. Let

$$\hat{x} = \lim_{j\to\infty} F(x_{n_j}),$$

then

$$F(\hat{x}) = \lim_{j\to\infty} F(x_{n_j+1}).$$

If $\hat{x} \neq F(\hat{x})$, there exist disjoint closed neighbourhoods U of $\hat{x}$ and V of $F(\hat{x})$. The mapping

$$\rho : U \times V \to \mathcal{R}, \quad \rho(x,y) = \frac{d(F(x), F(y))}{d(x,y)}$$

is continuous, and attains its maximum $k < 1$. Let $p \in \mathcal{N}$, such that for $j \geq p$

$$F(x_{n_j}) \in U, \quad F(x_{n_j+1}) \in V.$$

Then

$$d(F(x_{n_j+2}), F(x_{n_j+1})) \leq k.d(F(x_{n_j+1}), F(x_{n_j}))$$

and

$$d(F(x_n), F(x_{n+1})) \leq d(F(x_m), F(x_{m+1}))$$

for $n > m$.

Hence for $j > p$

$$\begin{aligned} d(F(x_{n_j}), F(x_{n_j+1})) &\leq d(F(x_{n_{j-1}+1}), F(x_{n_{j-1}+2})) \\ &\leq k.d(F(x_{n_{j-1}}), F(x_{n_{j-1}+1})) \leq \dots \\ &\leq k^{j-p+1}.d(F(x_{n_p+1}), F(x_{n_p+2})) \\ &\leq k^{j-p}.d(F(x_{n_p}), F(x_{n_p+1})). \end{aligned}$$

Therefore

$$d(\hat{x}, F(\hat{x})) = \lim_{j\to\infty} d(F(x_{n_j}), F(x_{n_j+1})) = 0.$$

This contradiction shows that $\hat{x}$ is a fixed point of F. The uniqueness follows from the contraction property. Finally, we will show $\hat{x} = \lim x_n$. This follows from

$$d(\hat{x}, F(x_{n_j+1})) = d(F^1(\hat{x}), F^1(F(x_{n_j}))) \leq d(\hat{x}, F(x_{n_j})).$$

□

The following example shows that there exist contractions on compact spaces, which are not strict:

Let $F : [-1,0] \to [-1,0]$ be defined by $F(x) = x + x^2$.

Then

$$\begin{aligned} F(x) - F(y) &= x + x^2 - y - y^2 = x - y + (x+y)(x-y) \\ &= (1 + x + y)(x - y). \end{aligned}$$

If $x \neq y$, then $|1 + x + y| < 1$, but there is no $k < 1$, such that $|F(x) - F(y)| \leq k|x - y|$.

As an application of Banach's fixed point theorem we will consider the following nonlinear **Volterra integral equation**

$$x(t) - \int_0^t k(t, \tau, x(\tau))d\tau = y(t). \qquad (*)$$

Assume the function

$$k : [0,1] \times [0,1] \times \mathcal{R} \to \mathcal{R}$$

is continuous and fulfills the following Lipschitz condition: there is a $\gamma > 0$, such that for all $t, \tau \in [0,1]$, $r, s \in \mathcal{R}$

$$|k(t,\tau,r) - k(t,\tau,s)| \leq \gamma |r - s|.$$

Then the mapping F, defined by

$$F(x)(t) = \int_0^t k(t, \tau, x(\tau))d\tau$$

maps $C[0,1]$ into $C[0,1]$. As the complete metric space we choose $(X, d) = (C[0,1], d_\gamma)$ with

$$d_\gamma(x_1, x_2) = \max_{0 \leq t \leq 1} |x_1(t) - x_2(t)| e^{-2\gamma}.$$

Then $F : X \to X$ and F is a $\frac{1}{2}$-contraction.

$$\begin{aligned} d_\gamma(F(x_1), F(x_2)) &= \max |F(x_1(t)) - F(x_2(t))| e^{-2\gamma} \\ &\leq \max \int_0^t |k(t,\tau,x_1(\tau)) - k(t,\tau,x_2(\tau))| d\tau . e^{-2\gamma t} \end{aligned}$$

$$
\begin{aligned}
&\leq \max \int_0^t \gamma|x_1(\tau) - x_2(\tau)|e^{-2\gamma\tau} d\tau . e^{-2\gamma t} \\
&\leq d_\gamma(x_1, x_2).\gamma. \max_{0\leq t\leq 1} \int_0^t e^{2\gamma\tau} d\tau . e^{-2\gamma t} \\
&\leq d_\gamma(x_1, x_2).\gamma. \frac{1}{2\gamma} \max e^{-2\gamma t}(e^{2\gamma t} - 1) \\
&\leq \frac{1}{2} d_\gamma(x_1, x_2).
\end{aligned}
$$

By Banach's fixed point theorem we have the following result.

The Volterra integral equation $(*)$ has for every continuous function y a unique continuous solution x.

Especially, we obtain the theorem of Picard - Lindelof :

The initial value problem

$$y' = f(t, y), \quad y(0) = \eta \qquad (\ *\ )$$

with Lipschitz continuous f has a unique solution, since $(\ *\ )$ is equivalent to

$$y(t) = \eta + \int_0^t f(\tau, y(\tau)) d\tau.$$

In general operator F is defined on a subset U of the complete metric space X or is a k-contraction only on a subset of X. In such cases it is an additional problem to find a subset $U_0 \subset U$ with the properties: F maps U_0 into U_0, and U_0 itself is a complete metric space. A sufficient condition for such a situation is described in the following result.

Theorem 1.4 *Let (X, d) be a complete metric space. Let $U \subset X$ and $F : U \to X$ be a k - contraction with $k < 1$. Let $x_1 \in U, x_2 = F(x_1) \in U, r = \frac{k}{1-k} d(x_1, x_2)$. Let the closed ball $U_0 := B(x_2, r) \subset U$. Then:*

1. *F maps U_0 into U_0.*

2. *F has a unique fixed point $\hat{x} \in U_0$ and the sequence $(x_n) = (F(x_{n-1}))$ converges to $\hat{x}$.*

Proof :

The closed subset $U_0 = B(x_2, r)$ is complete. Let $x \in U_0$. Then

$$\begin{aligned} d(F(x, x_2)) &= d(F(x), F(x_1)) \leq k.d(x, x_1) \\ &\leq k(d(x, x_2) + d(x_2, x_1)) \\ &\leq k(\frac{k}{1-k} + 1).d(x_1, x_2)) \\ &\leq r \end{aligned}$$

hence $F : U_0 \to U_0$ and Theorem 1.1 applies.

□

1.2 The Resolvent Operator

Let X be a Banach space and $U \subset X$. Let $F : U \to X$ be a continuous mapping. Let $V \subset X$ be a subset, such that for all $y \in V$ the equation

$$x - F(x) = y$$

has a unique solution x. Then x can be represented by

$$x = y - R(y)$$

and the mapping $R : V \to X$ is said to be the **resolvent operator** to F.

In the case, where F is a contraction with Lipschitz constant $k < 1$, the resolvent operator exists and is Lipschitz continuous, too.

Theorem 1.5 *Let X be a Banach space and $F : X \to X$ be Lipschitz continuous with Lipschitz constant $k < 1$. Then the resolvent operator R to F exists and has Lipschitz constant $\frac{k}{1-k}$.*

Proof :

For every $y \in X$ the equation

$$x - F(x) = y \tag{1.1}$$

has a unique solution, since the operator $F_0 : X \to X$ defined by

$$F_0(x) = F(x) + y$$

is a k-contraction and has a fixed point $x \in X$. So a mapping $G : X \to X$ with $G(y) = x$ is defined. Let $R(y) = y - G(y)$, then

$$x = y - R(y) \tag{1.2}$$

is the unique solution of equation (1.1). Let $y_1, y_2 \in X$ and $x_j - F(x_j) = y_j$. Then

$$\begin{aligned} \|R(y_1) - R(y_2)\| &= \|x_1 - y_1 - (x_2 - y_2)\| = \|F(x_1) - F(x_2)\| \\ &\leq k\|x_1 - x_2\| \\ &\leq k\|y_1 - R(y_2) - (y_2 - R(y_1))\| \\ &\leq k\|y_1 - y_2\| + k\|R(y_1) - R(y_2)\| \end{aligned}$$

hence

$$\|R(y_1) - R(y_2)\| \leq \frac{k}{1-k} \cdot \|y_1 - y_2\|$$

□

The representation

$$x = y - R(y)$$

of the solution x of equation (1.1) shows that properties of R determine the structure of the solution. We will illustrate this fact by the following result.

Theorem 1.6 *Let X be a Banach space, $F : U \to X$ a continuous operator and $R : V \to X$ its resolvent opeator. Let $Y \subset X$ be a linear subspace, such that* Range $F \subset Y$. *Then* Range $R \subset Y$.

Proof :

From equations (1.1) and (1.2) it follows that

$$R(y) = -F(x) = -F(y - R(y)).$$

Thus Range R $\subset$ Range $(-F) \subset Y$.

□

Theorem 1.7 *Let $\Omega \subset \mathcal{R}^n$ be an open subset, $X = C(\Omega)$. Let $k : \Omega \times \Omega \times \mathcal{R} \to \mathcal{R}$, such that $D_t^\alpha k(.,\tau,\xi)$ exists for all $\alpha \in \mathcal{N}^n$ with $|\alpha| \leq \nu, \tau \in \Omega, \xi \in \mathcal{R}$. Then*

$$F : C(\Omega) \to C(\Omega)$$

defined by

$$(Fx)(t) = \int_\Omega k(t,\tau,x(\tau))d\tau$$

maps $C(\Omega)$ into the subspaces $C^\nu(\Omega) \subset C(\Omega)$ of ν - times differentiable functions. If further the resolvent operator R to F exists, then the solution x of an equation

$$x - F(x) = y$$

$y \in C(\Omega)$, has the property $x - y \in C^\nu(\Omega)$.

1.3 The Theorem of the Local Homeomorphism

The strongest version of the solution of the equation

$$F(x) = y$$