EDITED BY
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THE PLASMA IN A MAGNETIC FIELD

The Plasma in a Magnetic Field

A Symposium on

MAGNETOHYDRODYNAMICS

Edited by

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PREFACE

The present volume is the second to grow out of a Lockheed-sponsored symposium on magnetohydrodynamics. That second symposium was held at the Palo Alto laboratory of the Missile Systems Division on December 16, 1957.

The general remarks of the Preface to the first volume concerning the intent of both the symposium and the book still apply. The changed title reflects the much wider interest in phenomena involving plasmas rather than liquid metals. Actually, all papers presented dealt with plasmas.

For large values of the temperature, electrons and ions travel quite long distances before the cumulative effect of collisions becomes appreciable. Some configurations of importance which are discussed in Section 1 can for this reason be studied best through an individual orbit analysis. Anyone who would rather avoid the complicated averaging procedure which relates particle orbits and electrical currents will welcome the derivation of a modified macroscopic theory which gets around contradictions with the individual orbit analysis.

The use of magnetic fields appears to be the most promising means for confining a hot deuterium plasma long enough to produce controlled thermonuclear power. However, the interface between a plasma and a magnetic field tends to be unstable. In Section 2 evidence is presented for such an instability from both pinch effect studies and astrophysical observations. It is cautioned that observed fusion reactions can and in some cases must be interpreted as being due to instabilities rather than to a high temperature. A discussion of the effectiveness of axial magnetic fields and conducting walls to stabilize a pinch is followed by a theoretical study of plasma confinement by radiation pressure.

The tight coupling between magnetic field and plasma makes it possible to transfer energy from one to the other. Section 3 shows how high-speed flow can be generated as well as modified by magnetic forces. Several arrangements for magnetically driving shock waves are presented. Effects which a magnetic field has on existing plasma flow, which are discussed in this section, include an observed channeling of a plasma stream passing through an axial magnetic field and the effect of a transverse field on drag and heat transfer deduced theoretically for simple viscous flow patterns.

I wish to thank the speakers at the symposium for their cooperation in preparing their papers for publication. I am also greatly indebted to D. Bershader for reading parts of the manuscript and to J. Todd from the LMSD publications staff for his valuable assistance in editing the manuscript and preparing it for publication.

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April 1958

ERRATA

Page 96, FIG. 9: To view these pictures correctly, turn the book 180 degrees. The caption then applies to the photographs in their new order.

Page 97, FIG. 10: To view these pictures correctly, follow the same procedure as above.

CONTENTS

SECTION 1

KINETIC THEORY

ADIABATIC INVARIANTS IN THE MOTIONS OF CHARGED PARTICLES

S. CHANDRASEKHAR*

I. INTRODUCTION

In current treatments of plasma physics one often considers the motions of charged particles in varying magnetic fields in a so-called guiding center approximation. In this approximation one separates the spiraling motion of the charged particles about the lines of force, from the motion along the lines of force. This separation of the motion into the two parts is possible only so long as the magnetic field remains sensibly constant, spatially, over several Larmor radii and, temporally, over several Larmor periods. When these latter conditions are fulfilled, one generally supposes that the transverse kinetic energy ($w_\perp$) of the spiraling motion divided by the strength of the magnetic field (B) remains constant during the motion. This constancy of

$$\mu = \frac{w_\perp}{B} = \frac{mv_\perp{}^2}{2B} \tag{1}$$

is not strictly an integral of the equations of motion; it is an *adiabatic invariant* in the sense that it is a constant in the limit of infinitely slow variation of the field.

If in virtue of the constancy of μ, the particle should be trapped between two regions of relatively strong field,† then one supposes that the integral

$$\oint v \cdot \frac{dB}{B} \tag{2}$$

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† If W ($= w_\perp + w_{||}$) denotes the total kinetic energy of the particle, then the points between which the particle will be trapped will be determined by $B_{\max} = W/\mu$.

of the component of the velocity parallel to the field taken over a complete cycle is a further adiabatic invariant.*

In this paper we shall examine the precise meaning which must be attached to the notion of adiabatic invariance with a view to clarifying the limitations in its use in plasma physics.

II. THE NOTION OF ADIABATIC INVARIANCE

The notion of adiabatic invariance played an important role in the early developments of the quantum theory in the context of formulating the general rules of quantization. Historically, it arose from a question proposed by Lorentz at the first Solvay Congress in 1911. Lorentz's question was: How does a simple pendulum behave when the length of the suspending thread is gradually shortened? The relevance of this question for the quantum theory of the time was the following: If an oscillator has originally the correct energy appropriate to an elementary quantum ($h\nu$), would the energy suffice to make up a quantum at the end of a process

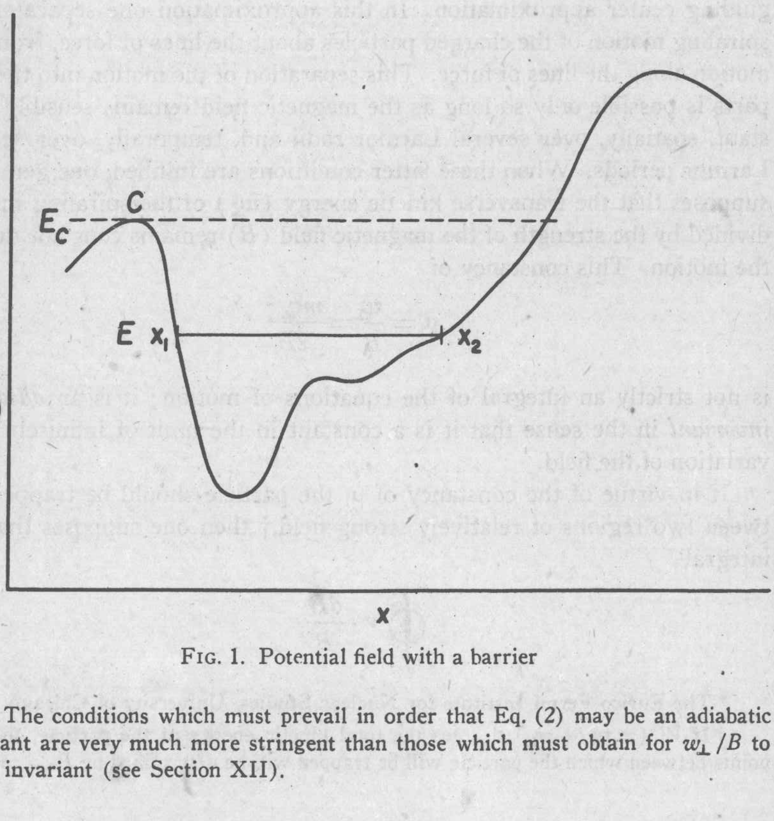

FIG. 1. Potential field with a barrier

* The conditions which must prevail in order that Eq. (2) may be an adiabatic invariant are very much more stringent than those which must obtain for $w_\perp/B$ to be an invariant (see Section XII).

(such as shortening the length of a pendulum) in which the frequency has been increased? To Lorentz's question, Einstein furnished the correct answer by saying that if the suspending thread is shortened infinitely slowly, then the energy, E, will increase proportionately to the "instantaneous" frequency, ν, so that a quantum of energy remains a quantum of energy; in other words, *E/ν is an adiabatic invariant.* We shall presently return to a somewhat more precise formulation of what this adiabatic invariance of E/ν really means; but it may be noted here, parenthetically, that the invariance of E/ν to slow changes was extended by Ehrenfest into a general "adiabatic hypothesis" applicable to multiply periodic systems. In this paper we shall, however, be concerned with adiabatic invariance as a concept in classical mechanics and, indeed, only for one-dimensional systems.

III. THE ADIABATIC INVARIANCE OF THE ACTION INTEGRAL

Consider the motion of a particle (of unit mass) in a potential field $V(x)$ (see Fig. 1). The equation governing its motion is

$$\frac{d^2x}{dt^2} + \frac{dV(x)}{dx} = 0. \tag{3}$$

This equation allows the energy integral

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E = \text{constant}; \tag{4}$$

from this we deduce that

$$t = \int \frac{dx}{\{2[E - V(x)]\}^{\frac{1}{2}}}. \tag{5}$$

If two points x_1 and x_2 exist such that (see Fig. 1)

$$E = V(x_1) = V(x_2) \qquad (x_2 > x_1), \tag{6}$$

then the particle will oscillate between x_1 and x_2; and the period of the oscillation will be given by

$$P = 2\int_{x_1}^{x_2} \frac{dx}{\{2[E - V(x)]\}^{\frac{1}{2}}} = \oint \frac{dx}{\{2[E - V(x)]\}^{\frac{1}{2}}}. \tag{7}$$

Associated with such periodic motions one defines the *action integral*

$$J = \oint dx\{2[E - V(x)]\}^{\frac{1}{2}} = \oint dx\left(\frac{dx}{dt}\right). \tag{8}$$

Suppose that $V(x)$ depends (apart from x) on a set of parameters a_j $(j=1,\ldots,n)$ such that

$$V(x) = V(x; a_1,\ldots, a_n). \tag{9}$$

Now let the a_j's instead of being constants be slowly varying functions of time. "Slowly varying" in this connection means that the particle executes several oscillations (appropriate to a set of a_j's which occur) before any of the a_j's change appreciably; in other words, we require:

$$\left|\frac{1}{a_j}\frac{da_j}{dt}\right| << \frac{1}{P(a_1,\ldots,a_n)} \qquad (j=1,\ldots,n). \tag{10}$$

When this is the case, we may characterize the motion by a certain energy, $\overline{E}$, appropriate for a single oscillation, namely,

$$\overline{E} = \oint \frac{dx}{\dot{x}}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x; a_1,\ldots,a_n)\right] \Big/ \oint \frac{dx}{\dot{x}}$$

$$= \oint dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x; a_1,\ldots,a_n)\right] \Big/ \oint dt. \tag{11}$$

It will be observed that in defining $\overline{E}$ in this way, we are averaging the energy during an oscillation, weighting each element of the orbit by the fraction of the time the particle spends in that element.

The theorem on adiabatic invariance is to the effect that *the action integral,*

$$J = \oint dx\,\{2[\overline{E} - V(x; a_1,\ldots,a_n)]\}^{\frac{1}{2}}, \tag{12}$$

defined in terms of $\overline{E}$ *is a constant for infinitely slow variations of the* a_j*'s.* The arguments by which one attempts to establish this constancy of J are, essentially, as follows:

We have

$$\frac{dJ}{dt} = \frac{\partial J}{\partial \overline{E}}\frac{d\overline{E}}{dt} + \sum_{j=1}^{n}\frac{\partial J}{\partial a_j}\frac{da_j}{dt}. \tag{13}$$

On the other hand, from the definition of J we have

$$\left.\begin{aligned} \frac{\partial J}{\partial a_j} &= -\oint \frac{\partial V}{\partial a_j} \frac{dx}{\{2\,[\overline{E} - V]\}^{\frac{1}{2}}} \\ \text{and} \\ \frac{\partial J}{\partial \overline{E}} &= \oint \frac{dx}{\{2\,[\overline{E} - V]\}^{\frac{1}{2}}} = \oint \frac{dx}{\dot{x}}; \end{aligned}\right\} \tag{14}$$

while from the definition of $\overline{E}$ we have

$$\frac{d\overline{E}}{dt} = \oint dt \left[\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x}\frac{dx}{dt} + \sum_{j=1}^{n} \frac{\partial V}{\partial a_j}\frac{da_j}{dt} \right] \Big/ \oint dt. \tag{15}$$

Making use of the equation of motion governing x, we can reduce the foregoing equation to the form

$$\begin{aligned} \frac{d\overline{E}}{dt} &= \sum_{j=1}^{n} \frac{da_j}{dt} \oint \frac{\partial V}{\partial a_j}\, dt \Big/ \oint dt \\ &= \sum_{j=1}^{n} \frac{da_j}{dt} \oint \frac{\partial V}{\partial a_j} \frac{dx}{\{2\,[\overline{E} - V]\}^{\frac{1}{2}}} \Big/ \oint \frac{dx}{\dot{x}}. \end{aligned} \tag{16}$$

Combining Eqs. (13), (14), and (16), we have

$$\begin{aligned} \frac{dJ}{dt} &= \sum_{j=1}^{n} \frac{da_j}{dt} \oint \frac{\partial V}{\partial a_j} \frac{dx}{\{2\,[\overline{E} - V]\}^{\frac{1}{2}}} \\ &\quad - \sum_{j=1}^{n} \frac{da_j}{dt} \oint \frac{\partial V}{\partial a_j} \frac{dx}{\{2\,[\overline{E} - V]\}^{\frac{1}{2}}} = 0, \end{aligned} \tag{17}$$

which establishes the constancy of J.

IV. ADIABATIC INVARIANCE IN THE SMALL AND IN THE LARGE

The proof of the constancy of the action integral given in Section III, for slow changes in the parameters which occur in the potential function, contains many heuristic elements. For example, the proof at no stage involved any explicit estimate of the error involved in the concept of $\overline{E}$ and of the action integral defined in terms of it. Indeed, the notion of adiabatic invariance was taken as equivalent to a justification of $\overline{E}$ and J as defined. This is not altogether satisfactory; and we shall accordingly try to formulate more explicitly the conceptual prerequisites underlying the notion of adiabatic invariance. In order that we may clarify the essential physical concepts without any of the formal complexities which a completely general formulation will require, we shall consider the relevant questions explicitly in the context of Lorentz's original problem and leave the generalizations to the reader.

Consider then the equation of motion of a simple pendulum. We have

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{18}$$

where ω denotes the circular frequency. If ω is a constant, the general solution of this equation can be written as

$$x = A \cos (\omega t + \varepsilon), \tag{19}$$

where A and ε are constants. From this solution it follows that

$$\langle \dot{x}^2 \rangle = \tfrac{1}{2} A^2\omega^2; \ \langle x^2 \rangle = \tfrac{1}{2} A^2; \tag{20}$$

and, further, that

$$\overline{E} = \tfrac{1}{2}\{\langle \dot{x}^2 \rangle + \omega^2 \langle x^2 \rangle\} = \tfrac{1}{2} A^2\omega^2. \tag{21}$$

The adiabatic invariance of the action integral as applied to this problem states the following:

Let ω, instead of being a constant, be a slowly varying function of time such that

$$\left| \frac{1}{\omega}\frac{d\omega}{dt} \right| << \omega. \tag{22}$$

For such slow variations

$$\overline{E}/\omega \text{ is a constant.} \tag{23}$$

If one wants to formulate the principle of adiabatic invariance without any of the heuristic elements implied in the forgeoing statement, then it would appear that the physical premises should be restated somewhat differently.

Clearly, the concepts of energy and period taken over from the case when ω is a constant cannot strictly apply to a dynamical system in which ω is some function of time no matter how slowly varying. They can apply only if certain limiting conditions are satisfied. Thus, let ω (t) be a function of time such that

$$\omega(t) \to \omega_1 \text{ as } t \to -\infty,$$

and (24)

$$\omega(t) \to \omega_2 \text{ as } t \to +\infty;$$

and also that

$$\frac{d^n\omega}{dt^n} \to 0 \quad \text{as } t \to +\infty \quad \text{and } t \to -\infty \quad \text{for all } n \geqslant 1. \tag{25}$$

When these conditions are satisfied, the system is a simple harmonic oscillator, in the strict sense, both when $t \to -\infty$ and when $t \to +\infty$. Accordingly, $\langle x^2 \rangle$ can be defined uniquely for both these limits and we may consider the ratio

$$\lambda = \frac{\omega_2 \langle x^2 \rangle_{t \to +\infty}}{\omega_1 \langle x^2 \rangle_{t \to -\infty}}. \tag{26}$$

This will clearly depend on the manner in which ω varies between ω_1 and ω_2. Let $d\omega/dt$ be bounded and

$$\text{Maximum of } \left| \frac{1}{\omega} \frac{d\omega}{dt} \right| = \frac{1}{T} \qquad (-\infty < t < +\infty). \tag{27}$$

A precise statement of the theorem on adiabatic invariance would be the assertion:

$$\lambda \to 1 \quad \text{as} \quad T \to \infty. \tag{28}$$

The remarkable aspect of this assertion is that it is not restricted by any limitation on ω_2/ω_1. Nevertheless, it will be convenient to distinguish two cases: the case when

$$\delta = \tfrac{1}{2} \left| \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} \right| << 1; \tag{29}$$

and when no such restriction applies. If, in the former case, we can show that

$$\lambda = 1 + O(\delta^2) \quad \text{as} \quad T \to \infty, \tag{30}$$

then we shall say that we have *adiabatic invariance in the small.* If on the other hand, Eq. (28) holds with no restriction on ω_2/ω_1, we shall say that we have *adiabatic invariance in the large.*

The principle of adiabatic invariance in the large can be formulated somewhat differently as follows: Let

$$\omega \equiv \omega(\alpha t) \tag{31}$$

represent a one-parameter family of time variations satisfying the requirements (24) and (25). For such a family of time variations, λ defined as in Eq. (26) will depend on α; and adiabatic invariance in the large implies that

$$\lambda(\omega_1, \omega_2; \alpha) \to 1 \quad \text{as} \quad \alpha \to 0, \tag{32}$$

independently of ω_1 and ω_2.

It is clear that the foregoing ideas formulated in the context of Lorentz's original problem can be extended to include the more general problem considered in Section III.

V. THE RELATION OF THE ADIABATIC INVARIANCE OF $w_\perp/B$ TO THE INVARIANCE OF $\omega\langle|x|^2\rangle$ IN THE PENDULUM PROBLEM

Before discussing in some detail the adiabatic invariance of $\omega\langle|x|^2\rangle$ in the problem of the simple pendulum, it will be useful to relate this problem to the invariance of $w_\perp/B$ in the motion of a charged particle in a varying magnetic field.

With the substitution

$$\zeta = x \exp\left(-i \int \omega\, dt\right), \tag{33}$$

the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{34}$$

becomes

$$\frac{d^2\zeta}{dt^2} + 2i\omega \frac{d\zeta}{dt} + i\frac{d\omega}{dt}\zeta = 0; \tag{35}$$

and this is the equation of motion of a charged particle in a spatially uniform but a temporally varying magnetic field, if we identify 2ω as the Larmor frequency ($= eB/mc$) and the real and the imaginary parts of ζ (regarded as a complex variable) as the Cartesian coordinates of the particle in a plane normal to the lines of force. The transverse kinetic energy divided by the instantaneous strength of the field is, apart from constant factors of proportionality, given by

$$\frac{1}{\omega}\left|\frac{d\zeta}{dt}\right|^2 = \frac{1}{\omega}\left|\frac{dx}{dt} - i\omega x\right|^2 \tag{36}$$