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Fourier Analysis and Partial Differential Equations

傅立叶分析和偏微分方程

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Dedicated to the memory of Tosio Kato

Preface

This book is the outcome of several courses and seminar talks held at the Instituto de Matemática Pura e Aplicada (IMPA) over the years. It is a greatly modified version of a previous work by the authors, *Equações Diferenciais Parciais, Uma Introdução*, (Projeto Euclides, IMPA, 1978). It has a twofold purpose, namely to introduce the student to the basic concepts of Fourier analysis and provide illustrations of recent applications where these concepts were used to study various properties of the solutions of some important nonlinear evolution equations.

The text is divided into three parts. The first one, containing Chapters 1 to 3, deals with Fourier series and periodic distributions. Chapters 4 to 6 belong to the second part, which contains applications of Fourier series and periodic distributions to partial differential equations. Chapters 7 and 8, in the third part, are more advanced and deal with some nonperiodic problems.

Chapter 1 presents some very classical material on PDEs, such as classification into types, separation of variables and maximum principles for the heat and Laplace equations. It is by no means a comprehensive account of such topics. Rather, it's purpose is to establish the basic language used throughout the work and to provide a collection of definitions and results needed in the remainder of the book. The following two chapters deal with Fourier series and some of its applications, first in a classical setting and then in the scenario provided by $\mathscr{P}'$, the space of periodic distributions. We include some general topological concepts that will be needed later on, and introduce L^2-type periodic Sobolev spaces using the fact that the Fourier transform is an isomorphism from $\mathscr{P}'$ onto the collection of all complex sequences of slow growth. In this way we reduce all our considerations to spaces of sequences and thus avoid the use of the Lebesgue integral until very late in the game. Chapter 4 concentrates on applications of the theory developed in the preceding chapters to linear evolution equations. We study a large number of such objects, including the heat, (free) Schrödinger and the wave equations. Although the chapter is interesting in its own right, its main purpose is to lay the groundwork for the applications to nonlinear evolution equations given in Chapters 5 and 6. We have included, for completeness' sake, a section summarizing the basic theory of semigroups of operators. Its purpose is to provide an abstract point of view for our treatment of linear evolution equations. It can be skipped without consequence to the understanding of the remainder of the text. Part Three addresses some situations that do not occur in the periodic setting, such as well-posedness in weighted Sobolev spaces and problems with initial conditions with 'infinite mass', that is, initial data that does not belong to $L^2(\mathbb{R})$. This is done in Chapter 8. Chapter 7 discusses the basic concepts of the theory of distributions, Sobolev spaces and presents applications to linear evolution equations, with emphasis on the heat and Schrödinger equations. Here we lay the groundwork for the applications studied in the final chapter. There are two appendixes. The first one summarizes the ODE theory used in the text while the second describes some technical commutator estimates needed to deal with the Korteweg–de Vries and related equations.

As is almost always the case, the choice of the topics discussed in this book is a direct consequence of the tastes and research interest of the authors. We have refrained from the study of classical elliptic theory, since there are many excellent works on the subject, and decided to concentrate on linear and nonlinear evolutions equations. As mentioned above, Lebegue's theory of integration is needed only very late in the book. The first point where it has to be used is in the proof of local-well-posedness of the Korteweg–de Vries equation presented in section 3 of Chapter 6. At that point we need Pettis' theorem on weakly measurable functions and the concept of absolutely continuous functions defined on an interval with values in a Banach space. However, the reader who is unfamiliar with these ideas may skip the section, because all the relevant results are stated in Sections 1 and 2. In Part Three, it is no longer feasible to avoid the theory of integration, and there we assume that the reader is familiar with the essentials of the theory as presented in the books by Bartle, Royden or Rudin, mentioned in the bibliography. We emphasize that our avoidance of using Lebesgue integration is intended to make most of the book available to advanced undergraduates or beginning graduates that are still unfamiliar with the theory. In fact, familiarity with integration theory would enhance the understanding of the book, and make it more pleasurable to read.

A final word about prerequisites. We assume that the reader is familiar with the material usually covered in functional analysis courses, up to the theory of compact operators. We also assume familiarity with the basic theory of ordinary differential equations, more specifically with the results presented in Appendix B.

Finally the authors wish to thank our friends Carlos Augusto Isnard (IMPA), Felipe Linares (IMPA) and Marcia Scialom (IMECC\UNICAMP) for several interesting conversations on the subject matter of this book and for reading various parts of the original manuscript. It goes without saying that any mistakes found in the text are the authors', and only the authors', responsibility. And last, but not least, our thanks to the long suffering and patient David Tranah, of Cambridge University Press, who gave us all the support we needed while writing this book.

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Contents

Part One

Fourier Series and Periodic Distributions

1

Preliminaries

In this chapter we present some basic definitions and some of the problems and concepts that will be discussed and used throughout the book. The material presented here is by no means a complete account of such topics as classification into types, canonical forms, the method of characteristics and so on. There are several excellent accounts of these in the literature (see [57], [60], [64], [86] and [151] for example).

1.1 Basic Definitions and Examples

Let us begin by introducing some notation and terminology. An **open ball** of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$ is a set of the form

$$B(x_0;r) = \{x \in \mathbb{R}^n : |x - x_0| < r\},$$

where x_0 is a fixed point in $\mathbb{R}^n$, $|\cdot|$ is the usual euclidean norm in $\mathbb{R}^n$ and r is a positive real number. Similarly, a **closed ball** in $\mathbb{R}^n$ is a set of the form $\overline{B(x_0;r)} = \{x \in \mathbb{R}^n : |x - x_0| \leqslant r\}$. A subset $\Omega \subseteq \mathbb{R}^n$ is said to be **open** if, for any $x \in \Omega$, there exists an open ball $B(x;r)$ contained in Ω. A subset $K \subseteq \mathbb{R}^n$ is **closed** if its complement $\mathbb{R}^n \setminus K = \{x \in \mathbb{R}^n : x \notin K\}$ is open. The **closure** of $S \subseteq \mathbb{R}^n$, denoted by $\overline{S}$, is the smallest closed set containing S, i.e., $\overline{S} = \bigcap\{K \subseteq \mathbb{R}^n : K$ is closed and $S \subseteq K\}$. The **interior** of $S \subseteq \mathbb{R}^n$, denoted by $\mathrm{Int}(S)$, is the largest open set contained in S, that is, $\mathrm{Int}(S) = \bigcup\{\Omega \subseteq \mathbb{R}^n : \Omega$ is open and $\Omega \subseteq S\}$. The **boundary** of $S \subseteq \mathbb{R}^n$ is the set $\partial S = \overline{S} \cap \overline{(\mathbb{R}^n \setminus S)}$. It is easy to see that the closed ball $\overline{B(x_0;r)}$ is in fact the closure of the open ball $B(x_0;r)$, that the interior of the closed ball $\overline{B(x_0;r)}$ is the open ball $B(x_0;r)$ and that the boundary of both the open and the closed balls is the **sphere** $\{x \in \mathbb{R}^n : |x - x_0| = r\}$. An open subset $\Omega \subseteq \mathbb{R}^n$ is **connected** if there are no disjoint nonempty open sets Ω_1 and Ω_2 in $\mathbb{R}^n$ such that $\Omega = \Omega_1 \cup \Omega_2$. An open connected subset of

$\mathbb{R}^n$ is called a **domain**. As usual, if $\Omega \subseteq \mathbb{R}^n$ is an open subset, we denote by $C^k(\Omega)$ the set of all functions $\Omega \to \mathbb{C}$ that are k times continuously differentiable. The **support** of a function $f : \Omega \longrightarrow \mathbb{C}$, denoted by *supp(f)*, is the smallest closed set outside which f vanishes identically. We use the notation $C_0^k(\Omega)$ for the set of all functions $\Omega \to \mathbb{C}$ that are k times continuously differentiable and have compact support in Ω. The set of all complex valued infinitely differentiable functions on Ω is denoted by $C^\infty(\Omega)$ and the set of all complex valued infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. If $[a,b] \subseteq \mathbb{R}$ is a closed interval, $C^k([a,b])$ is the set of all functions $f : [a,b] \to \mathbb{C}$ that are k times differentiable in the closed interval with the kth derivative $f^{(k)} \in C([a,b])$; the differentiability at the endpoints is defined by the one-sided limits

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h},$$
$$f'(b) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h}.$$

We define in a similar way $C^k([a,\infty))$ and $C^k((-\infty,b])$, where $a,b \in \mathbb{R}$. For infinitely differentiable functions we will use the notations $C^\infty([a,b])$, $C^\infty([a,\infty))$ and $C^\infty((-\infty,b])$.

A **differential equation (DE)** is an equation involving one or more independent variables, an unknown function, and its derivatives with respect to these variables. If there is only one independent variable x, we say that the equation is an **ordinary differential equation (ODE).** If there are two or more independent variables, $x_1, x_2, \dots, x_n$, we say that the equation is a **partial differential equation (PDE).** Thus, an ODE is an expression of the form

$$F\left(x, u, u', \dots, u^{(m)}\right) = 0 \tag{1.1}$$

where $u', \dots, u^{(m)}$ denote the derivatives of $u(x)$ with respect to x up to order m in some open subset of $\mathbb{R}$, while a PDE has the form

$$G\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^m u}{\partial x_n^m}\right) = 0, \tag{1.2}$$

where $x = (x_1, x_2, \dots, x_n)$ belongs to some open set $\Omega \subseteq \mathbb{R}^n$, F and G are given functions, u is to be determined and

$$\frac{\partial^j u}{\partial x_1^{j_1} \partial x_2^{j_2} \dots \partial x_n^{j_n}}, \; j = j_1 + j_2 + \dots + j_n,$$

denotes the jth order partial derivative of u. We will often use the following alternative notations.

$$\partial_{x_j}^k u = \frac{\partial^k u}{\partial x_j^k} = u_{\underbrace{x_j \ldots x_j}_{k \text{ times}}}.$$

The above definitions are too general. It is easy to devise very strange and useless equations like

$$\exp\left(u'(x)\right) = 0$$

or

$$\frac{1}{(u'(x))^2 + u(x)} = 0.$$

Thus, it is important to determine which equations are meaningful and restrict one's attention to those subclasses. In the remainder of this section we will exhibit several examples of interesting equations that will be considered in the course of the book.

The **order** of a partial differential equation is the order of the highest order derivative occurring in the equation. If F and G are not constant, when considered as functions of the derivatives of order m, then both (1.1) and (1.2) have order m. A partial differential equation is **linear** if it is a polynomial of the first degree in u and its derivatives. Otherwise, the PDE is **nonlinear.** The most general second order linear PDE has the form

$$\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + c(x) u(x) + d(x) = 0, \tag{1.3}$$

where at least one of the coefficients $a_{jk}(x)$ is not identically zero. If $d = 0$, we say that (1.3) is **homogeneous;** otherwise (1.3) is **nonhomogeneous.** The **principal part** of a PDE is the part of the equation that contains the derivatives of highest order. In the case of (1.3), the principal part is the double sum on the left hand side. Nonlinear equations with linear principal parts are called **semilinear.** The general second order semilinear PDE is

$$\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} = f\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right).$$

The most important examples of linear equations are the following.

EXAMPLE 1.1. The three classical equations of mathematical physics, as follows.

- The **heat equation**

$$\partial_t u(t,x) = \alpha^2 \Delta u(t,x), \ t>0, x \in \Omega \subseteq \mathbb{R}^n, \tag{1.4}$$

where α^2 is a constant, known as the **diffusion coefficient**, and

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \tag{1.5}$$

is the **Laplacian** (or **Laplace operator**) in $\mathbb{R}^n$. This equation is associated with diffusion phenomena, like the flow of heat in a conducting medium (see [60], [114], [151] and [162]).

- The **wave equation**

$$\partial_t^2 u(t,x) = c^2 \Delta u(t,x), \ t>0, x \in \Omega \subseteq \mathbb{R}^n. \tag{1.6}$$

This equation describes wave phenomena, like the motion of a membrane or waves traveling in a string. Here c is a positive constant, known as the **speed of propagation** of the wave (see [60], [114], [125] and [151]).

- **Laplace's equation**

$$\Delta u(x) = 0, \ x \in \Omega \subseteq \mathbb{R}^n. \tag{1.7}$$

This equation describes stationary phenomena, such as the electrostatic potential generated by fixed distributions of electric charges (see [60], [83], [114], [151] and [155], for example). Note that the stationary (i.e. time independent) solutions of the heat and wave equations satisfy Laplace's equation. Functions satisfying (1.7) are said to be **harmonic** in Ω.

EXAMPLE 1.2. The nonhomogeneous versions of the equations in Example 1.1, that is,

$$\partial_t u(t,x) = \alpha^2 \Delta u(t,x) + f(t,x), \ t>0, x \in \Omega \subseteq \mathbb{R}^n, \tag{1.8}$$

$$\partial_t^2 u(t,x) = c^2 \Delta u(t,x) + g(t,x), \ t>0, x \in \Omega \subseteq \mathbb{R}^n, \tag{1.9}$$

$$\Delta u(x) = h(x), \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.10}$$

where f, g, and h are given functions. Equation (1.10) is known as **Poisson's equation**.

EXAMPLE 1.3. **Schrödinger's equation**

$$i\partial_t u(t,x) = -\frac{\hbar}{2m} \Delta u(t,x) + V(x) u(t,x), \ t>0, x \in \Omega \subseteq \mathbb{R}^n, \tag{1.11}$$