LINEAR ALGEBRA AND PROJECTIVE GEOMETRY

Linear Algebra and Projective Geometry

BY REINHOLD BAER

University of Illinois, Urbana, Illinois

1952

ACADEMIC PRESS INC., PUBLISHERS

NEW YORK, N.Y.

PREFACE

In this book we intend to establish the essential structural identity of projective geometry and linear algebra. It has, of course, long been realized that these two disciplines are identical. The evidence substantiating this statement is contained in a number of theorems showing that certain geometrical concepts may be represented in algebraic fashion. However, it is rather difficult to locate these fundamental existence theorems in the literature in spite of their importance and great usefulness. The core of our discussion will consequently be formed by theorems of just this type. These are concerned with the representation of projective geometries by linear manifolds, of projectivities by semi-linear transformations, of collineations by linear transformations and of dualities by semi-bilinear forms. These theorems will lead us to a reconstruction of the geometry which was the starting point of our discourse within such (apparently) purely algebraic structures as the endomorphism ring of the underlying linear manifold or the full linear group.

Dimensional restrictions will be imposed only where they are necessary for the validity of the theorem under consideration. It is, for instance, well known that most of these existence theorems cease to be true if the dimension is too low. Thus we will have to exclude the low dimensions quite often. But finiteness of dimension will have to be assumed only in exceptional cases; and this will lead us to a group of finiteness criteria. Similarly we will obtain quite a collection of criteria for the commutativity of the field of scalars; the Index lists all of them. Only the characteristic two will be treated in rather a cavalier fashion, being excluded from our discussion whenever it threatens to be inconvenient.

From the remarks in the last paragraph it is apparent that certain topics ordinarily connected with linear algebra cannot appear in our presentation. Determinants are ruled out, since the existence of determinants enjoying all the desirable properties implies commutativity of the field of scalars. Matrices will make only fleeting appearances, mainly to show that they really have no place in our discussion. The invariant concept is after all that of linear transformation or bilinear form and any choice of a representative matrix would mean an inconvenient and unjustifiable fixing of a not-at-all distinguished system of coordinates.

All considerations of continuity have been excluded from our discussion in spite of the rather fascinating possibilities arising from the interplay of algebraic and topological concepts. But the founders of projective geometry conceived it as the theory of intersection and joining, purely algebraic concepts. Thus we felt justified in restricting our discussion to topics of an algebraic nature and to show how far one may go by purely algebraic methods.

Some sections have been labeled "Appendix" since the topics treated in them are not needed for the main body of our discussion. In these appendices either we discuss applications to special problems of particular interest or we investigate special situations of the general theory in which deeper results may be obtained. No subsequent use will be made of these so the reader may omit them at his discretion.

Little actual knowledge is presupposed. We expect the reader to be familiar with the basic concepts and terms of algebra like group, field, or homomorphism, but the facts needed will usually be derived in the form in which we are going to use them. Ample use will be made of the methods of transfinite set theory—no metaphysical prejudice could deter the author from following the only way to a complete understanding of the situation. For the convenience of the reader not familiar with this theory we have collected the concepts and principles that we need in a special appendix at the end of the book. No proofs are given in this appendix; for these the reader is referred to the literature.

No formal exercises are suggested anywhere in the book. But many facts are stated without proof. To supply the missing arguments will give the reader sufficient opportunity to test his skill.

The references are designed almost exclusively to supply "supplementary reading," rounding off what has been said or supplying what has been left unsaid. We have not tried to trace every concept and result to its origin. What we present here is essentially the combined achievement of a generation of algebraists who derived their inspiration from Dedekind, Hilbert, and Emmy Noether; what little the author may have added to the work of his predecessors will presumably be clear to the expert.

We turn finally to the pleasant task of thanking those who helped us: the editors of this series, in particular Professor S. Eilenberg who read a draft of the manuscript and gave freely of his advice; Professors Eckmann and Nakayama and Dr. Wolfson who helped with reading the proofs; and last but not least my wife who drew all the figures and read all the proofs. The publishers and their staff helped us greatly and we are extremely obliged to them for the way they treated our wishes concerning the book's make-up.

REINHOLD BAER

February 1952

TABLE OF CONTENTS

CHAPTER I

Motivation

The objective of this introductory chapter is to put well-known geometrical facts and concepts into a form more suitable to the ways of present day algebraical thinking. In this way we shall obtain some basic connections between geometrical and algebraical structures and concepts that may serve as justification and motivation for the fundamental concepts: linear manifold and its lattice of subspaces which we are going to introduce in the next chapter. All the other concepts will be derived from these; and when introducing these derived concepts we shall motivate them by considerations based on the discussion of this introductory chapter.

Since what we are going to do in this chapter is done only for the purposes of illustration and connection of less familiar concepts with such parts of mathematics as are part of everybody's experience, we shall choose for discussion geometrical structures which are as special as is compatible with our purposes. Reading of this chapter might be omitted by all those who are already familiar with the essential identity of linear algebra and affine and projective geometry. We add a list of works which elaborate this point.

A Short Bibliography of Introductory Works Emphasizing the Mutual Interdependence of Linear Algebra and Geometry

L. Bieberbach: Analytische Geometrie. Leipzig, Berlin, 1930.

L. Bieberbach: Projektive Geometrie. Leipzig, Berlin, 1930.

G. Birkhoff and S. MacLane: A Survey of Modern Algebra. New York, 1948.

W. Blaschke: Projektive Geometrie. Wolfenbüttel, 1948.

P. Halmos: Finite Dimensional Vector Spaces. *Ann. Math. Studies* 7. Princeton, N. J., 1940.

L. Heffter and C. Köhler: Lehrbuch der analytischen Geometrie, Bd. 1, 2. Karlsruhe, Leipzig, 1929.

W. W. D. Hodge and D. Pedoe: Methods of Algebraic Geometry. Cambridge, 1947.

C. C. MacDuffee: Vectors and Matrices. Carus Mathematical Monographs 7. Ithaca, N. Y., 1943.

O. Schreier and E. Sperner: Einführung in die analytische Geometrie. Bd. 1, 2. Leipzig, Berlin, 1931-1935.

O. Schreier and E. Sperner: Introduction to Modern Algebra and Matrix Theory. Translated by M. Davis and M. Hausner. New York, 1951.

B. Segre: Lezioni di geometria moderna. Vol. I: Fondamenti di geometria sopra un corpo qualsiasi. Bologna, 1948.

I.1. The Three-Dimensional Affine Space as Prototype of Linear Manifolds

The three-dimensional real affine space may be defined as the totality E [$= E_3$] of triplets (x,y,z) of real numbers x,y,z. This definition is certainly short, but it has the grave disadvantage of giving preference to a definite system of coordinates, a defect that will be removed in due course of time.

The triplets (x,y,z) are usually called the points of this space. Apart from these points we shall have to consider lines and planes, but we shall not discuss such concepts as distance or angular measurement as we want to adhere to the affine point of view. It is customary to define a plane as the totality of points (x,y,z) satisfying a linear equation

$$xa + yb + zc + d = 0$$

where a,b,c,d are real numbers and where at least one of the numbers a,b,c is different from 0; and a line may then be defined as the intersection of two different but intersecting planes. It is known that the points on a line as those on a plane may be represented in the soc. parametric form; and we find it more convenient to make these parametric representations the starting point of our discussion.

The points of a line L may be represented in the form:

$$L: \begin{cases} x = tu + a \\ y = tv + b \\ z = tw + c \end{cases}$$

where (a,b,c) is some point on the line L, where (u,v,w) is a triplet of real numbers, not all 0, and where the parameter t ranges over all the real numbers. As t ranges over all the real numbers, $(tu + a, tv + b, tw + c)$ ranges over all the points of the line L. To obtain a concise notation for this we let $P = (a,b,c)$ and $D = (u,v,w)$, and then we put

$$(tu + a, tv + b, tw + c) = tD + P.$$

Algebraically we have used, and introduced, two operations: the addition of triplets according to the rule

$$(x,y,z) + (x',y',z') = (x + x', y + y', z + z');$$

and the (scalar) multiplication of a triplet by a real number according to the rule

$$t(x,y,z) = (tx,ty,tz).$$

Indicating by $R(x,y,z)$ the totality of triplets of the form $t(x,y,z)$, we may denote the totality of points on the line L by

$$L = RD + P$$

where we have identified the line L with the set of its points.

Using the operations already introduced we may now treat planes in a similar fashion. Consider three triplets $P = (a,b,c)$, $D' = (u',v',w')$ and $D'' = (u'',v'',w'')$. Then the totality N of points of the form:

$$t'D' + t''D'' + P$$

where the parameters t' and t'' may range independently of each other over all the real numbers may be designated by

$$N = RD' + RD'' + P.$$

If both D' and D'' are the 0-triplet $[D' = D'' = (0,0,0) = 0]$, then N degenerates into the point P; if D' or D'' is 0 whereas not both D' and D'' are 0, then N degenerates into a line. More generally N will be a line whenever D' is a multiple of D'' or D'' is a multiple of D' [and not both are 0]. But if N is neither a point nor a line, then N is actually the totality of points on a plane; or as we shall say more shortly: N is a plane.

In this treatment of lines and planes we have considered the line $L = RD + P$ as the line through the two points P and $P + D$ and the plane $N = RD' + RD'' + P$ as the plane spanned by the three not collinear points P, $P + D'$, $P + D''$. The question arises under which circumstances two pairs of points determine the same line, or two triplets of points span the same plane, and more generally how to characterize by internal properties of the set those sets of points which form a line or a plane.

With this in mind we introduce the following

Definition: The not vacuous set S of points in E is a *flock* of points, if $sU - sV + W$ belongs to S whenever s is a real number and U, V, W are in S.

Note that

$$s(u,u',u'') - s(v,v',v'') + (w,w',w'') = (su - sv + w, su' - sv' + w', su'' - sv'' + w'').$$

A set consisting of one point only certainly has this property; and the reader will find it easy to verify that lines and planes too are flocks of points. Trivially the totality of points in E is a flock. Consider now conversely some flock S of points. This flock contains at least one point P. If P is the only point in S, then we have finished our argument. Assume therefore that S contains a second point Q. It follows from the flock property that S contains the whole line

$$L = P + R(P - Q)$$

—note that $P - Q \neq (0,0,0)$. If this line exhausts S, then we have again reached our goal; and thus we may assume that S contains a further point K, not on L. It follows from the flock property that S contains the totality

$$N = P + R(Q - P) + R(K - P);$$

and N is a plane, since K is not on L. If N exhausts S, then again we have achieved our end. If, however, there exists a point M in S, but not in N, then one may prove that $S = E$ [by realizing that the four points P,Q,K,M are "linearly independent," and that therefore every further point "depends" on them]; we leave the details to the reader.

Now we may exhibit those features of the space E which are "coordinate-free." The space E consists of elements, called points. These points may be added and subtracted $[P \pm Q]$ and they form an additive abelian group with respect to addition. There exists furthermore a scalar multiplication rP of real numbers r by points P with the properties:

$$(r + s)P = rP + sP, r(P + Q) = rP + rQ, (rs)P = r(sP), 1P = P.$$

There exist furthermore distinguished sets of points, called flocks in the preceding discussion; they are characterized by the closure property:

If U, V, W are in the flock F, and if r is a real number, then $rU - rV + W$ belongs to F.

Affine geometry may then be defined (in a somewhat preliminary fashion) as the study of the flocks in the space E.

Among the flocks those are of special interest which contain the origin (the null element with respect to the addition of points). It is easy to see that a set S of points is a flock containing the origin if, and only if,

(a) S contains $P + Q$ and $P - Q$ whenever S contains P and Q, and

(b) S contains rP whenever r is a real number and P is a point in S.

In other words the flocks through the origin are exactly the subsets of E which are closed under addition, subtraction and multiplication or, as we shall always say, the flocks through the origin are exactly the subspaces of E.

If T is a subspace of E (closed under addition, subtraction and multiplication) and if P is a point, then $T + P$ is a flock. If S is a flock, then the totality T of points of the form $P - Q$ for P and Q in S is a subspace, and S has the form $S = T + P$ for some P in S. This shows that we know all the flocks once we know the subspaces; and so in a way it may suffice to investigate the subspaces of E. But the observation has been made that the totality of lines and planes through the origin of a three-dimensional affine space has essentially the same structure as the real projective plane; and this remark we want to substantiate in the next section.

I.2. The Real Projective Plane as Prototype of the Lattice of Subspaces of a Linear Manifold

We begin by stating the following definition of the real projective plane which has the advantage of being short and in accordance with customary terminology, but has the disadvantage of giving preference to a particular system of coordinates.

Every triplet (x_0,x_1,x_2) of real numbers, not all 0, represents a point, and every point may be represented in this fashion.

The triplets (x_0,x_1,x_2) and (y_0,y_1,y_2) represent the same point if, and only if, there exists a number $c \neq 0$ such that $x_i = cy_i$ for $i = 0,1,2$.

Every triplet (u_0,u_1,u_2) of real numbers, not all 0, represents a line, and every line may be represented in this fashion.

The triplets $(u_0,u_1,u_2,)$ and (v_0,v_1,v_2) represent the same line if, and only if, there exists a number $d \neq 0$ such that $u_i = v_i d$ for $i = 0,1,2$.

The point represented by (x_0,x_1,x_2) is on the line represented by (u_0,u_1,u_2) if, and only if,

$$x_0u_0 + x_1u_1 + x_2u_2 = 0.$$

If we use notations similar to those used in I.1, then we may say that the triplets x and y represent the same point if, and only if, $x = cy$, and that the triplets u and v represent the same line if, and only if, $u = vd$. The principal reason for writing the scalar factor d on the right will become apparent much later [II.3]; at present we can only say that some of our formulas will look a little better. If we define the scalar product of the triplets x and u by the formula

$$xu = \sum_{i=0}^{2} x_i u_i,$$

then the incidence relation "point x on line u" is defined by $xu = 0$. We note that $xu = 0$ implies $(cx)u = 0$ and $x(ud) = 0$.

If x is a triplet, not 0, then the totality of triplets cx with $c \neq 0$ represents the same point, and thus we may say without any danger of confusion that Rx is a point. Likewise uR may be termed a line whenever u is a triplet not 0.

If Rx is a point, then this is a set of triplets closed under addition and multiplication by real numbers. If uR is a line, then we may consider the totality S of triplets x such that $xu = 0$. It is clear that S too is closed under addition and multiplication by real numbers; and that S is composed of all the points Rx on the line uR. Thus we might identify the line uR with the totality S.

But now we ought to remember that the totality of triplets $x = (x_0, x_1, x_2)$ is exactly the three-dimensional affine space discussed in I.1, and that the points Rx and the lines S, discussed in the preceding paragraph, are just what we called in I.1 subspaces of the three-dimensional affine space. That all subspaces—apart from 0 and E—are just points and lines in this projective sense, the reader will be able to verify without too much trouble. Once he has done this, he will realize the validity of the contention we made at the end of I.1:

The real projective plane is essentially the same as the system of subspaces (= flocks through the origin) of the three-dimensional real affine space.

Consequently all our algebraical discussion of linear manifolds admits of two essentially different geometrical interpretations: the affine interpretation where the elements (often called vectors) are the basic atoms of discussion and the projective interpretation where the subspaces are the elementary particles. We shall make use of both interpretations feeling free to use whichever is the more suitable one in a special situation, but in general we shall give preference to projective ways of thinking.

The real affine space and the real projective plane are just two particularly interesting members in a family of structures which may be obtained from these special structures by generalization in two directions: first, all limitations as to the number of dimensions will be dropped so that the dimension of the spaces under consideration will be permitted to take any finite and infinite value (though sometimes we will have to exclude the very low dimensions from our discussion); secondly we will substitute for the reals as field of coordinates any field whatsoever whether finite or infinite, whether commutative or not. But in all these generalizations the reader will be wise to keep in mind the geometrical picture which we tried to indicate in this introductory chapter.

CHAPTER II

The Basic Properties of a Linear Manifold

In this chapter the foundations will be laid for all the following investigations. The concepts introduced here and the theorems derived from them will be used almost continuously. Thus we prove the principle of complementation and the existence of a basis which contains a basis of a given subspace; we show that any two bases contain the same number of elements which number (finite or infinite) is the rank of the space. It is then trivial to derive the fundamental rank identities, which contain as a special case the theory of systems of homogeneous linear equations, as we show in Appendix I, and to relate the rank of a space with the rank of its adjoint space (= space of hyperplanes).

II.1. Dedekind's Law and the Principle of Complementation

A *linear manifold* is a pair (F, A) consisting of a (not necessarily commutative) field F and an additive abelian group A such that the elements in F operate on the elements in A in a way subject to the following rules:

(a) If f is an element in F and a an element in A, then their product fa is a uniquely determined element in A.

(b) $(f' + f'')a = f'a + f''a$, $f(a' + a'') = fa' + fa''$ for f, f', f'' in F and a, a', a'' in A.

(c) $1a = a$ for every a in A [where 1 designates the identity element in F].

(d) $(f'f'')a = f'(f''a)$ for f', f'' in F and a in A.

From these rules one deduces readily such further rules as

(e) $0a = f0 = 0$ for f in F and a in A [where the first 0 is the null element in F whereas the second and third 0 stand for the null element in A];

(f) $(-f)a = f(-a) = -(fa)$ for f in F and a in A.

REMARK ON TERMINOLOGY: It should be noted that we use the word "field" here in exactly the same fashion as other authors use terms like division ring, skew field, and sfield. Thus a field is a system of at least two elements with two compositions, addition and multiplication. With respect to addition the field is a commutative group; the elements, not 0, in the field form a group, which need not be commutative, with respect to multiplication; and addition and multiplication are connected by the distributive laws. A good example of a field which is not commutative is provided by the real quaternions; see, for instance, Birkhoff-MacLane [1], p. 211 for a discussion.

In a linear manifold we have two basic classes of elements: those in the additive group A (the vectors) and those in the field F (the scalars). To keep these two classes of elements apart it will sometimes prove convenient to refer to the elements in the field F as to "numbers in F," a terminology that seems to be justified by the fact that numbers in F may be added, subtracted, multiplied and divided.

Instead of linear manifold we shall use expressions like F-space A, etc; and we shall often say that F is the field of coordinates of the space A. Note that in the literature also terms like F-group A, F-modulus A, vector space A over F are used.

A *linear submanifold* or *subspace* of (F,A) is a non-vacuous subset S of A meeting the following requirements:

(g) $s'-s''$ belongs to S whenever s',s'' are in S; and fs belongs to S whenever s is in S and f in F.

If we indicate as usual by $X+Y$ and $X-Y$ respectively the sets of all the sums $x+y$ and $x-y$ with x in X and y in Y, and by GX the totality of products gx for g in G and x in X, then one sees easily the equivalence of (g) with the following conditions:

$$S = S + S = S - S = FS.$$

We note a few simple examples of such linear submanifolds: 0; the points Fp with $p \neq 0$; the lines $Fp + Fq$ [where Fp and Fq are distinct points]; the planes $L + Fp$ where L is a line and Fp is a point, not part of L. A justification for these terms, apart from the reasons already given in Chapter I, will be given in the next section.

Instead of linear submanifold we may also use terms like subspace, F-subgroup and admissible subspace.

Our principal objective is the study of the totality of subspaces of a given linear manifold. This totality has a certain structure, since subspaces are connected by a number of relations.

Containedness or Inclusion: If S and T are subspaces, and if every element in S belongs to T, then we write $S \leqslant T$ and say that S is part of T or S is on T or S is contained in T. If $S \leqslant T$, but $S \neq T$, then we write $S < T$. If H contains K and K contains L, then K may be said to be "between" H and L.

INTERSECTION: If S and T are subspaces, then $S \cap T$ is the set of all the elements which belong to both S and T. It is readily seen that $S \cap T$ is a subspace too.

If Φ is a set of subspaces, then we define as the intersection of the subspaces in Φ the set of all the elements which belong to each of the subspaces in Φ. This intersection is again a subspace, and will be indicated in a variety of ways. For instance, if Φ consists only of a finite number of subspaces $S_1, \cdots, S_n$, then their intersection will be written as $S_1 \cap \cdots \cap S_n$; if the subspaces in Φ are indicated by subscripts: $\Phi = [\cdots S_\sigma \cdots]$, then we write the intersection as $\bigcap_\sigma S_\sigma$ and so on.

Instead of intersection the term cross cut is used too.

SUM. If S and T are subspaces, then their sum $S + T$ consists of all the elements $s + t$ with s in S and t in T. One verifies easily that $S + T$ too is a subspace, the subspace "spanned" by S and T.

If $S_1, \cdots, S_n$ are a finite number of subspaces, then their sum

$$S_1 + \cdots + S_n = \sum_{i=1}^{n} S_i$$

consists of all the sums $s_1 + \cdots + s_n = \sum_{i=1}^{n} s_i$ with s_i in S_i. Again it is clear that the sum of the S_i is a subspace, the subspace spanned by the S_i.

If finally Φ is any set of subspaces, then their sum consists of all the sums $s_1 + \cdots + s_k$ where each s_i belongs to some subspace S in Φ. This again is a subspace which may be indicated in a variety of ways, like $\sum_\nu S_\nu$. Note that only finite sums of elements in A may be formed, though we may form the sum of an infinity of subspaces. Furthermore it should be verified that the definition of the sum of a finite number of subspaces is a special case of our definition of the sum of the subspaces in Φ.

Intersection and sum of subspaces are connected by the following rule which is easily verified:

The sum of the subspaces in Φ is the intersection of all the subspaces which contain every subspace in Φ; the intersection of the subspaces in Φ is the sum of all the subspaces which are contained in every subspace in Φ.

We turn now to the derivation of more fundamental relations.

Dedekind's Law: *If R, S, T are subspaces, and if $R \leqslant S$, then*

$$S \cap (R + T) = R + (S \cap T).$$

PROOF: From $S \cap T \leqslant S$ and $R \leqslant R + T$ and $R \leqslant S$ we deduce

$$R + (S \cap T) \leqslant S \cap (R + T).$$

If conversely the element s belongs to $S \cap (R + T)$, then $s = r + t$ with r in R and t in T. From $R \leqslant S$ we infer now that $s - r$ belongs to S. Hence $t = s - r$ belongs to $S \cap T$ so that $s = r + t$ belongs to $R + (S \cap T)$. Consequently $S \cap (R + T) \leqslant R + (S \cap T)$; and this proves the desired equation.

The reader should construct examples which show that the above equation fails to hold without the hypothesis $R \leqslant S$.

Two more concepts are needed for the enunciation of the next law.

Quotient Spaces: If M is a subspace, then we define congruence modulo M by the following rule:

The elements x and y in A are congruent modulo M, in symbols: $x \equiv y$ modulo M, if their difference $x - y$ belongs to M.

One verifies that congruence modulo M is reflexive, symmetric, and transitive; and thus we may divide A into mutually exclusive classes of congruent elements. Congruences may be added and subtracted, since $x \equiv y$ modulo M and $x' \equiv y'$ modulo M imply $x + x' \equiv y + y'$ modulo M and $x - x' \equiv y - y'$ modulo M. Since for every f in F we may deduce from $x \equiv y$ modulo M the congruence $fx \equiv fy$ modulo M, we may multiply congruences by elements in F.

Complete classes of congruent elements modulo M are often called cosets modulo M. The totality of these cosets modulo M we designate by A/M. Addition and subtraction of cosets modulo M is defined by the corresponding operations with the elements in the cosets; and the product fX of f in F by X in A/M is just the totality of all fx for x in X, unless $f = 0$ in which case we let $fX = 0X = M = 0$. Then it is clear from the preceding discussion that $(F, A/M)$ is likewise a linear manifold. It may be said that the F-space A/M arises from the F-space A by substituting congruence modulo M for the original equality.

If the subspace S of A contains M, then the elements in S form complete classes of [modulo M] congruent elements. We may form S/M; and one sees easily that S/M is a subspace of A/M.

Conversely let T be some subspace of A/M. Every element in T is a class of congruent elements in A; and thus we may form the set T^* of all the elements in A which belong to some class of congruent elements in T. One verifies that T^* is a subspace of A which contains M and which satisfies $T^*/M = T$.

The reader ought to discuss the example where A is the real projective plane and M some point in it. Then the subspaces of A/M correspond essentially to the lines of A which pass through the point M; and their totality has "the structure of a line."

An Isomorphism of the F-Space A upon the F-Space B is a one-to-one correspondence σ mapping the elements in A upon the elements in B in such a way that

$$A\sigma = B, (a+b)\sigma = a\sigma + b\sigma, (fa)\sigma = f(a\sigma)$$

for a,b in A and f in F.—It is clear that the inverse σ^{-1} may be formed, and that σ^{-1} is an isomorphism of B upon A.

This concept of isomorphism may be applied in particular upon subspaces and their quotient spaces.

The existence of an isomorphism between the F-spaces A and B we indicate by saying that A and B are isomorphic and by writing $A \sim B$. Instead of isomorphism we are going to say usually "*linear transformation.*" Thus linear transformation signifies what in classical terminology is called a non-singular linear transformation. The concept of isomorphism is going to be extended later when we introduce the more comprehensive concept of semi-linear transformation [III.1].

Isomorphism Law: *If S and T are subspaces of the F-space A, then* $(S+T)/S \sim T/(S \cap T)$.

PROOF: Every element x in $S+T$ has the form $x = s + t$ with s in S and t in T. Clearly $x \equiv t$ modulo S. Thus every element X in $(S+T)/S$ contains elements in T; and we may form the non-vacuous intersection $X \cap T$ of the sets X and T. If x' and x'' belong both to $X \cap T$, then $x' \equiv x''$ modulo S so that $x' - x''$ belongs to $S \cap T$; and now one verifies that $X \cap T$ is an element in $T/(S \cap T)$.

If Y is an element in $T/(S \cap T)$, then $S + Y$ is easily seen to be an element in $(S+T)/S$. Since

$$X = S + (X \cap T) \quad \text{for } X \text{ in } (S+T)/S,$$
$$Y = T \cap (S + Y) \quad \text{for } Y \text{ in } T/(S \cap T),$$

we see that the mappings: $X \rightarrow X \cap T$ and $Y \rightarrow S + Y$ are reciprocal mappings between $T/(S \cap T)$ and $(S+T)/S$; and thus they are in particular one-to-one correspondences between the two quotient spaces. That they are actually isomorphisms is now quite easily verified (so that we may leave the verification to the reader). This completes the proof of the Isomorphism Law.

Lemma: *The join of an ordered set of subspaces is a subspace.*

PROOF: If Φ is an ordered set of subspaces of A, and if $S \neq T$ are distinct subspaces in Φ, then one and only one of the relations $S < T$ and $T < S$ is valid. Denote by J the join of all the subspaces in Φ so that an element belongs to J if, and only if, it belongs to at least one subspace in Φ. If x and y are elements in J, then there exist subspaces X and Y in Φ such that x is in X and y in Y. It follows from our hypothesis that one of these subspaces contains the other one, say $X \leqq Y$. Then x and y and consequently $x - y$ are in Y so that $x - y$ belongs to J. Since x is in X, so