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V. I. Arnol'd (Ed.)

Dynamical Systems VIII

Singularity Theory II: Applications

动力系统 VIII

奇异理论 II：应用

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

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R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Institute for Scientific Information (VINITI), ul. Usievicha 20a, 125219 Moscow, Russia

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V.I. Arnol'd, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Russia

Authors

V.I. Arnol'd, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Russia
V.V. Goryunov, Moscow Aviation Institute, Volokolamskoe sh. 4, 125871 Moscow, Russia
O.V. Lyashko, Airforce Engineering Academy, Leningradskij pr. 40, 125167 Moscow, Russia
V.A. Vasil'ev, Institute of Applied Mathematics of the Russian Academy of Sciences, Miusskaya pl. 4, 125047 Moscow, Russia

Translator

J.S. Joel, P.O. Box 132, Kelly, WY 83011, USA

Singularity Theory II
Classification and Applications

V.I. Arnol'd, V.V. Goryunov, O.V. Lyashko, V.A. Vasil'ev

Translated from the Russian
by J.S. Joel

Contents

Foreword

In the first volume of this survey (Arnol'd et al. (1988), hereafter cited as "EMS 6") we acquainted the reader with the basic concepts and methods of the theory of singularities of smooth mappings and functions. This theory has numerous applications in mathematics and physics; here we begin describing these applications. Nevertheless the present volume is essentially independent of the first one: all of the concepts of singularity theory that we use are introduced in the course of the presentation, and references to EMS 6 are confined to the citation of technical results.

Although our main goal is the presentation of an already formulated theory, the reader will also come upon some comparatively recent results, apparently unknown even to specialists. We point out some of these results.

In the consideration of mappings from $\mathbb{C}^2$ into $\mathbb{C}^3$ in § 3.6 of Chapter 1, we define the bifurcation diagram of such a mapping, formulate a $K(\pi, 1)$-theorem for the complements to the bifurcation diagrams of simple singularities, give the definition of the Mond invariant N in the spirit of "hunting for invariants", and we draw the reader's attention to a method of constructing the image of a mapping from the corresponding function on a manifold with boundary. In § 4.6 of the same chapter we introduce the concept of a versal deformation of a function with a nonisolated singularity in the class of functions whose critical sets are arbitrary complete intersections of fixed dimension. The corresponding $K(\pi, 1)$-theorem for simple functions is also true here.

In Chapter 2, following V.I. Bakhtin, we discuss the topology of four variants of the Maxwell variety of the singularity A_5, we describe generic "perestroikas" of the maximum function under a change of one parameter of the four and apply this analysis to the study of perestroikas of shock waves propagating in a three-dimensional space (following I.A. Bogaevskiĭ); here we give a "sec + tan" formula for the number of components of the space of Morse functions on the line and a formula of A.A. Vakulenko for the number of components of complements to Maxwell strata.

Chapter 3 contains, among other things, classifications of the singularities of the boundary of the set of hyperbolic differential equations (B.Z. Shapiro and A.D. Vaĭnshteĭn) and the singularities of the boundary of the set of fundamental systems of solutions of linear differential equations. (This theory, due to M.È. Kazaryan, is related to the Schubert stratification of a Grassmannian, to the bifurcation of Weierstrass points of algebraic curves, and to the theory of the focal varieties of projective curves.) In the same chapter we discuss the singularities of the boundary of the set of disconjugate systems (i.e., Chebyshev systems)—the connection of this question with the Schubert stratifications of flag varieties and with Bruhat orderings was recently discovered by B.Z. and M.Z. Shapiro.

Historically the first result based on the theory of monodromy is Newton's theorem on the nonintegrability of plane ovals; in § 4.1 we shall prove multidimensional generalizations of this theorem and give some new Picard-Lefschetz formulas that arise naturally in this problem. § 4.2 is devoted to the theory of Petrovskiĭ lacunas, studying the regularity of the fundamental solutions of hyperbolic partial differential equations close to wave fronts. Among other things, here we shall prove a converse to Petrovskiĭ's local criterion for hyperbolic operators in general position.

In Chapter 5 we enumerate the local lacunas (domains of regularity) for many of the singularities of wave fronts that appear in the tables, including all simple singularities and all singularities of corank 2 with Milnor number $\leqslant 11$. A significant part of these lacunas were found using a computer algorithm, which enumerates all the nonsingular morsifications of complicated real singularities; in § 5.3 we describe this algorithm.

The references inside the volume are organized in the following way. If a reference is to some place within the same chapter, then we give the number of the appropriate section or subsection, as in the Table of Contents. If the reference is to another chapter, then the number of the chapter appears before the number of the section or subsection. References to the first volume are kept to a minimum and are indicated as references to "EMS 6".

Chapter 1 was written by V.V. Goryunov, Chapters 2 and 3, except for § 3.5, were written by V.I. Arnol'd, and Chapters 4 and 5 were written by V.A. Vasil'ev. § 3.5 was written by B.Z. Shapiro. The authors offer their sincere thanks to him.

Chapter 1
Classification of Functions and Mappings

In this chapter we consider classifications with respect to the most frequently encountered and naturally occurring equivalence groups. The principal objects of attention here are the simple singularities, and also the topology of the non-singular fiber of a mapping and the geometry of bifurcation diagrams. Many of the properties that we shall discuss are analogous to the properties of functions with isolated critical points that were presented in EMS 6 (Arnol'd et al. (1988)). In order to make the presentation as independent as possible from EMS 6, we shall, in the appropriate places, recall the definitions and constructions introduced in EMS 6 for isolated singularities of functions and carried over to those singularities that we treat in this chapter.

Important branches of the theory of singularities such as equivariant mappings remain outside of our consideration. We intend to devote a separate paper in one of the later volumes of this series to them.

§ 1. Functions on a Manifold with Boundary

A *manifold with boundary* is a smooth (real or complex) manifold with a fixed hypersurface. Two functions on a manifold with boundary are said to be *equivalent* if they are mapped into each other under a diffeomorphism of the manifold that maps the boundary into itself. The classification of functions on a manifold with boundary is closely connected with the Lie groups B_k, C_k, F_4, and G_2, and the Coxeter groups H_3, H_4 and $I_2(p)$, whose Dynkin diagrams have multiple edges (Bourbaki (1968)). This connection is analogous to the connection that occurs between the groups A_k, D_k, and E_k and the singularities of functions on smooth manifolds without boundary (EMS 6, 2.5).

1.1. Classification of Functions on a Manifold with a Smooth Boundary. Recall that a function or a mapping is said to be *simple* with respect to some equivalence group if, by an arbitrary sufficiently small perturbation of it, we can obtain representatives of only a finite number of equivalence classes. Thus, for the equivalence of germs of functions $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with respect to the group of coordinate changes in the source (so-called $\mathcal{R}$- or right-equivalence) the simple functions are precisely those that, for a suitable choice of coordinates, have the following normal forms:

$A_k, k \geqslant 0$	$D_k, k \geqslant 4$	E_6	E_7	E_8
$\pm y_1^{k+1}$	$y_1^2 y_2 \pm y_2^{k-1}$	$y_1^3 \pm y_2^4$	$y_1^3 + y_1 y_2^3$	$y_1^3 + y_2^5$

This list is given up to stable equivalence (two *functions* of different numbers of variables are said to be *stably equivalent* if they become equivalent after addition of nondegenerate quadratic forms in the extra variables).

Consider a manifold with a smooth boundary. Locally this is the germ at zero of the pair $(\mathbb{R}^n, \mathbb{R}^{n-1})$ or $(\mathbb{C}^n, \mathbb{C}^{n-1})$.

Theorem (Arnol'd (1978)). *The germs of simple functions at a boundary point of a real manifold with a smooth boundary are described completely, up to a diffeomorphism in the source that takes the boundary into itself, by the following list of germs of functions $f(x, y)$ at the point $x = 0$, $y = 0$ of the boundary $x = 0$:*

$A_k, k \geqslant 0$	$D_k, k \geqslant 4$	E_6	E_7	E_8
$\pm y^{k+1} + x$	$y_1^2 y_2 \pm y_2^{k-1} + x$	$y_1^3 \pm y_2^4 + x$	$y_1^3 + y_1 y_2^3 + x$	$y_1^3 + y_2^5 + x$

$B_k, k \geqslant 2$	$C_k, k \geqslant 3$	F_4
$\pm x^k \pm y^2$	$xy \pm y^k$	$\pm x^2 + y^3$

When we talk about equivalence of functions that do not have the same number of variables here we mean stable equivalence on a manifold with boundary.

Remark. The set of nonsimple functions has codimension $n + 3$ in the space of functions that take the value 0 at $0 \in \mathbb{R}^n$.

Consider the complex situation. We pass from the manifold $\mathbb{C}^n$ with boundary $x = 0$ to a two-sheeted covering of it, branched along the boundary, by setting $x = z^2$ and $y = y$. There is a natural involution $(z, y) \mapsto (-z, y)$ on the covering. A germ of the function $f(x, y)$ on the manifold with boundary corresponds to a germ of $f(z^2, y)$, which is invariant under the involution. In this way we obtain a one-to-one correspondence between functions on a manifold with a smooth boundary and functions that are invariant under an involution of the space $\mathbb{C}^n$ that preserves the subspace $\mathbb{C}^{n-1}$. This one-to-one correspondence is also a one-to-one correspondence of equivalence classes of singularities. Therefore, the preceding theorem also gives a classification of the simple functions that are invariant under the above action of the group $\mathbb{Z}_2$.

Later research has shown that the above list arises either wholly or partially in many other classification problems. These problems include the projections onto the line and mappings of the plane into three-space described in § 3, and also the linear singularities from § 4.

Recall that the *modality* of a point of a manifold on which a Lie group acts is the smallest number m such that any sufficiently small neighborhood of this point intersects only a finite number of at most m-parameter families of orbits of the group action. The points of modality 0 are therefore precisely the simple points. The points of modality 1 and 2 are respectively called *unimodal* and *bimodal.*

The classification of functions on a manifold with a smooth boundary $x = 0$, which do not have critical points on the ambient space, is easily seen to be equivalent to the classification of the restrictions of these functions to the boundary. The normal forms of such functions are obtained by adding the function x to the normal form of the restriction (cf. the singularities A_k, D_k and E_k in the absolute and boundary variants). In comparison with Chapter 1 of EMS 6 the essentially new point in the classification of boundary singularities is therefore just the classification of functions that have a critical point on the ambient manifold. Up to stable equivalence such functions of modality 1 are exhausted by the following two lists (for the definition of the number μ see Subsection 1.2) (Arnol'd (1978), Matov (1981a, 1981b)).

Unimodal boundary singularities of corank 2

Notation	$\mathbb{C}$-normal form	Restrictions	μ
$F_{1,0}$	$x^3 + axy^2 + y^3$	$4a^3 + 27 \neq 0$	6
$F_{1,p}$	$ax^{p+1} + xy^2 + y^3$	$a \neq 0, p \geqslant 1$	$6 + p$
F_8	$x^4 + y^3 + ax^3y$	—	8
F_9	$x^3y + y^3 + ax^2y^2$	—	9
F_{10}	$x^5 + y^3 + ax^4y$	—	10
$K_{4,2}$	$y^4 + axy^2 + x^2$	$a^2 \neq 4$	6
$K_{4,q}$	$y^4 + axy^2 + x^q$	$a \neq 0, q > 2$	$q + 4$
$K_{p,q}$	$y^p + xy^2 + ax^q$	$a \neq 0, p > 4, q \geqslant 2$	$p + q$
$K^{\#}_{1,2p-3}$	$(x + y^2)^2 + ax^py$	$a \neq 0, p > 1$	$2p + 3$
$K^{\#}_{1,2p-4}$	$(x + y^2)^2 + ax^p$	$a \neq 0, p > 2$	$2p + 2$
K^*_8	$y^4 + x^2y + ax^3$	—	8
K^*_9	$y^4 + x^3 + ax^2y^2$	—	9
K^{**}_8	$y^5 + x^2 + axy^3$	—	8

Unimodal boundary singularities of corank 3

Notation	$\mathbb{R}$-normal form	Restrictions	μ
$L_6 = D_{4,1}$	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	$a^2 \pm 1 \neq 0$	6
$D_{k,l}$	$y_1^2y_2 \pm y_2^{k-1} + axy_1^l + xy_2$	$a \neq 0, k \geqslant 4, l \geqslant 1, k + l > 5$	$k + l + 1$
$E_{6,0}$	$y_1^3 \pm y_2^4 + axy_1 + xy_2$	—	8
$E_{7,0}$	$y_1^3 + y_1y_2^3 + axy_1 + xy_2$	—	9
$E_{8,0}$	$y_1^3 + y_2^5 + axy_1 + xy_2$	—	10
D_5^1	$y_1^2y_2 \pm y_2^4 + xy_1 + axy_2^2$	—	8
$E_{6,1}$	$y_1^3 \pm y_2^4 + xy_1 + axy_2^2$	—	9
D_4^2	$y_1^2y_2 \pm y_2^3 \pm x^2 \pm axy_1^2$	—	8

The bimodal boundary singularities have been classified by Matov (1981b).

Definition. A class of singularities X *adjoins* or is *adjacent* to a class of singularities Y ($X \to Y$) if every function of the class X can be deformed into a function of the class Y by an arbitrarily small perturbation.

All the adjunctions of simple boundary singularities are given in Fig. 1, and the most important adjunctions of unimodal boundary singularities are given in Fig. 2.

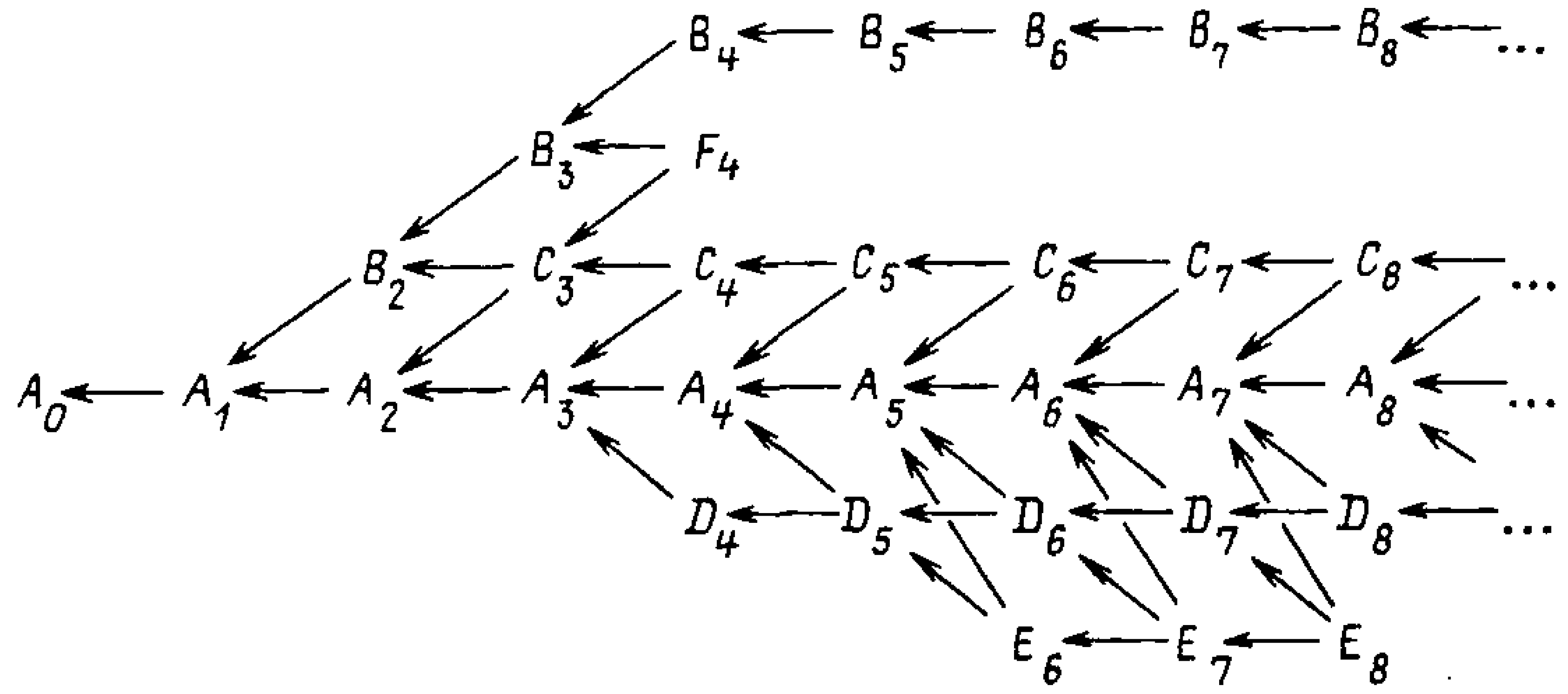

Fig. 1. Adjacent simple boundary singularities

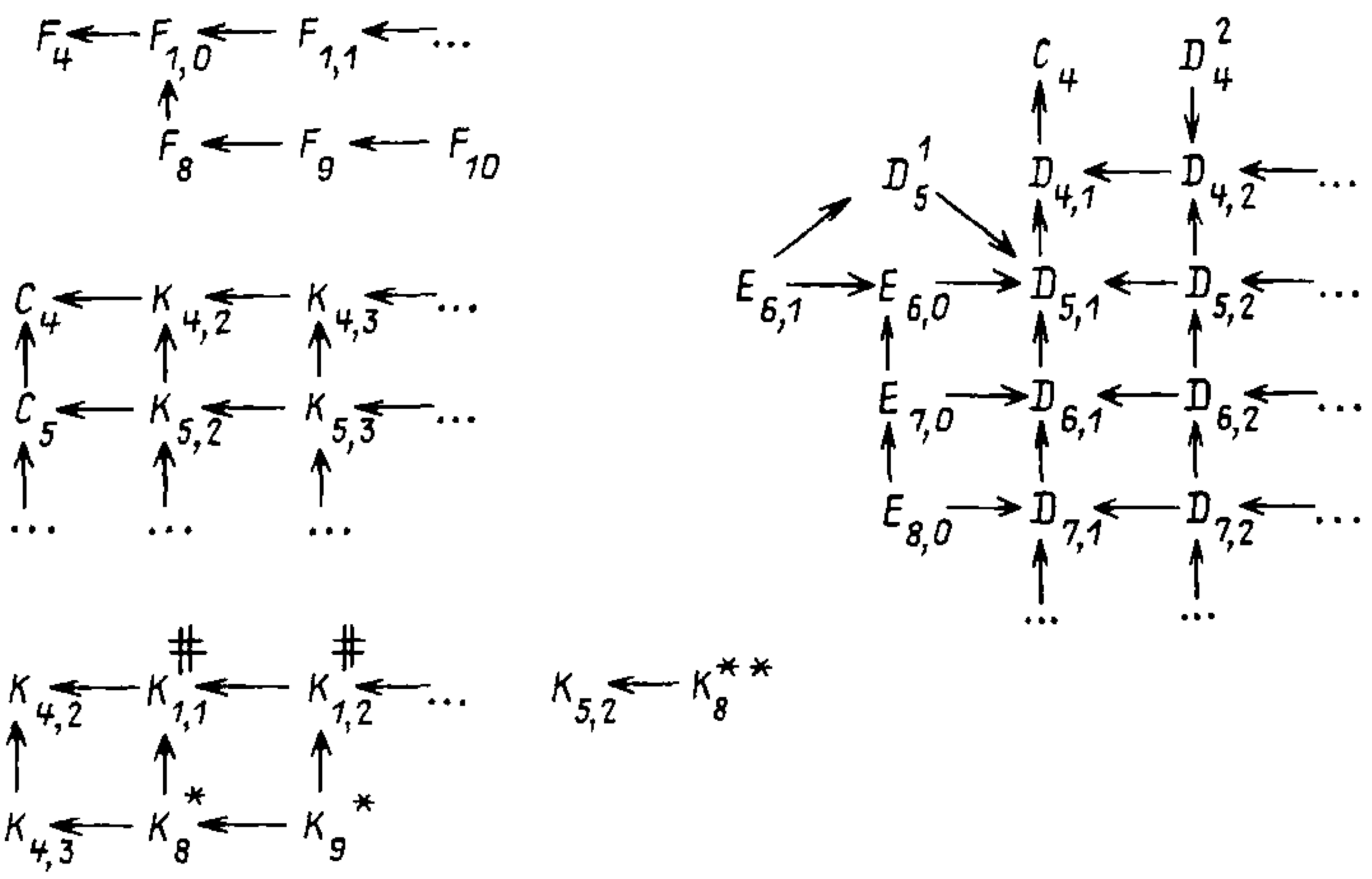

Fig. 2. Main adjunctions of unimodular boundary singularities

1.2. Versal Deformations and Bifurcation Diagrams. We begin by recalling some definitions that relate to deformations of functions and mappings.

A *deformation* of an element f of a manifold M is a germ of a smooth mapping $F\colon (\Lambda, 0) \to (M, f)$ of a finite-dimensional linear space Λ (the *base* of the deformation). The deformation *induced* from f under a mapping $\theta\colon (\Lambda', 0) \to (\Lambda, 0)$ is the deformation $\theta^*F = F \circ \theta$.

Suppose that a Lie group G acts on M. Two deformations F_1 and F_2 of the same element f with a common base Λ are said to be *equivalent* if there exists a deformation $g\colon (\Lambda, 0) \to (G, e)$ of the identity of G such that

$$F_1(\lambda) = g(\lambda)F_2(\lambda).$$

Finally, a deformation F of the point f is said to be *versal* if every other deformation of this point is equivalent to a deformation induced from F. A versal deformation whose base has the smallest possible dimension is termed *miniversal.*

If the manifold M is finite-dimensional, then a versal deformation of a point is a germ of a transversal to the orbit of the point. This result is also valid for points of infinite-dimensional spaces of mappings, having orbits of finite codimension, in the case when the group action is sufficiently good (see EMS 6, 3.2).

For complex boundary singularities we can take the space $\mathcal{O}_n$ of germs at the origin of holomorphic functions on $\mathbb{C}^n$ as the manifold. As a Lie group acting on this set we can take the pseudogroup of germs of diffeomorphisms of $\mathbb{C}^n$ that preserve the boundary $x = 0$. In this case a miniversal deformation of the germ $f(x, y_1, \ldots, y_{n-1})$ from $\mathcal{O}_n$ with $f(0) = 0$ is given by a transversal to the orbit:

$$F(x, y, \lambda) = f(x, y) + \lambda_1 e_1(x, y) + \cdots + \lambda_\mu e_\mu(x, y),$$

where the λ_i are the parameters of the deformation and $e_1, \ldots, e_\mu$ is a basis of the local ring

$$Q_f = \mathcal{O}_n/\mathcal{O}_n\langle xf_0, f_1, \ldots, f_{n-1}\rangle, \quad f_0 = \partial f/\partial x, \quad f_j = \partial f/\partial y_j, \quad j > 0.$$

The number $\mu = \dim_{\mathbb{C}} Q_f$ is called the *multiplicity of the critical point* 0. The germs of functions with critical points of infinite multiplicity form a set of infinite codimension in the space of germs.

For simple functions μ is the subscript in the notation of the singularity.

The number μ is related to the Milnor number of the function f on the ambient space and to that of the restriction of f to the boundary:

$$\mu = \mu_1 + \mu_0,$$
$$\mu_1 = \dim_{\mathbb{C}} \mathcal{O}_n/\mathcal{O}_n\langle f_0, f_1, \ldots, f_{n-1}\rangle$$
$$\mu_0 = \dim_{\mathbb{C}} \mathcal{O}_{n-1}/\mathcal{O}_{n-1}\langle f_1', \ldots, f_{n-1}'\rangle, \quad f' = f|_{x=0}.$$

We fix a representative of the versal deformation F and choose a sufficiently small ball $B_\rho \subset \mathbb{C}^n$ of radius ρ with center at the origin. We choose a δ that is sufficiently small with respect to ρ and for λ from the ball $|\lambda| < \delta$ we consider the local level set $V_\lambda = \{(x, y) \in B_\rho : F(x, y, \lambda) = 0\}$.

Definition. The local *level set* V_λ is said to be *nonsingular* if 1) 0 is not a critical value for $F(\cdot, \lambda)$, and 2) the manifold V_λ is transversal to the boundary.

The germ at zero of the hypersurface $\Sigma \subset \mathbb{C}^\mu$ consisting of those values of λ for which the set V_λ is singular is called the *bifurcation diagram of the zeros* (*discriminant*) *of the function* f. The discriminant has two components Σ_1 and Σ_2 corresponding to level manifolds that are not smooth and to those that are not transversal to the boundary. Of course, for functions that are not critical on the ambient space, the first component is empty and the discriminant coincides with the usual discriminant of its restriction to the boundary (see EMS 6, 1.1.10). Figure 3 shows the discriminant of the singularity C_3. The discriminant of B_3 looks the same, except that the components Σ_1 and Σ_0 change places.