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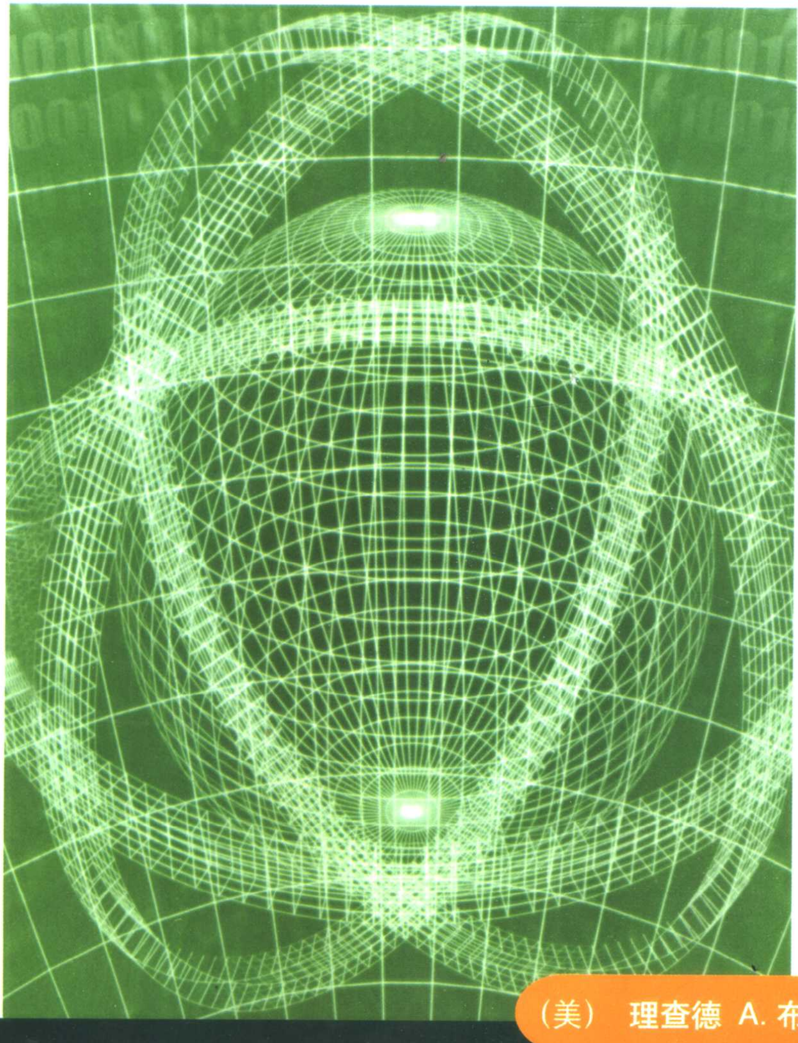
(英文版 · 第4版)



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(美) 理查德 A. 布鲁迪 著

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前 言

在第3版的前言中曾经提到如何重写某些章节以及如何添加某些新的材料和练习。从第2版到第3版一些主要的变化如下：

第4章添加了偏序和等价关系的介绍。

第5章增加了再论偏序集一节，其中证明了Dilworth定理及其对偶。

第8章加写了正整数分拆的新内容。

第11章是本书讨论图论的第一章，其中，树被定义为移去任意一边后都不再连通的连通图，并删掉了介绍有向图的一节。

第12章是新的一章，讨论有向图和网络。这一章包括Ford和Fulkerson的最大流最小割定理的证明，由此，第9章的Menger定理和König定理作为推论而导出。

第2版第12章讨论的图论中基本的数构成第3版的第13章。Pólya计数法原先在第13章，后来改成第14章。

第4版修正了我所知道的所有排印错误，从语言上做了一些小的调整(包括在论述图论的各章中用“路径”代替“链”)，插入某些零星内容，并增加了60多道新的具有挑战性的练习题。我不愿把这本书改变过多或是加入太多新的课题，也不喜欢有过多词汇的书籍(而本前言就没有太多的词汇)，不想陷进那样的陷阱。此外，本版添加两节新内容。第6章添加新的最后一节，论述莫比乌斯反演，作为容斥原理的推广。第8章新增了一节，论述格路径以及小Schröder数和大Schröder数。

如同早期版本一样，可以使用本书作为一学期或两学期的本科课程。第一学期可侧重于计数法，而第二学期则侧重图论和设计。也可以合在一起作为一学期的课程，讨论某些计数法和图论，或者讨论一些计数法和设计理论。下面是对每章的简短评述和各章间相互关系的介绍：

第1章是引论性的一章；我通常从这章选择一两个课题并最多花费两节课讲述这一章。第2章讨论鸽巢原理，至少应该以缩减的形式讨论。不过要注意，这对于后面鸽巢原理某些困难的应用以及Ramsey定理的理解却无济于事。第3章到第8章主要讨论计数技术和计数结果序列的某些性质。应该按顺序讨论它们。第4章涉及排列和组合的生成方法，还有上面提到的偏序和等价关系的介绍。然而，除第5章论述偏序集的那一节外，第4章后面各章基本上都与第4章无关。因此，第4章可以略去或压缩，甚至根本不讨论偏序集。第5章论述二项式系数的性质，第6章讨论容斥原理，论述莫比乌斯反演的新的一节在后面各章都用不到。第7章比较长，讨论递推关系的求解以及计数中生成函数的使用。第8章主要关于Catalan数、第一类和第二类Stirling数、分拆数以及小Schröder数和大Schröder数。后面各章与第8章无关。

第9章讨论二分图中的匹配问题。虽然本书在介绍图之前引入了二分图，但本章与后面的图论各章基本上没有什么关系。除匹配理论对拉丁方的应用外，论述设计的第10章独立于本书其余部分。不过，在10.4节末尾用到第9章建立的匹配理论。第11章和第13章对图论进行了广泛的

讨论，重点放在图论算法上。第12章涉及有向图和网络流。第14章处理在置换群作用下的计数问题，这里确实用到了先前许多的计数思想。除最后一个例子外，本章与图论和设计的各章无关。

当教授本书一学期的课程时，我喜欢以论述Pólya计数法的第14章作为结束，这能够使我们解决许多计数问题，而这些问题用前面各章的方法是无法解决的。在第14章之后，给出了本书大约650道练习题中的部分解答和提示。一些练习旁边标有星号“*”，表明它们更具有挑战性。在证明的最后和例子的末尾均标有符号“□”以示结束。

确定阅读本书的前提条件比较困难。如同所有想要作为教材的著作一样，高度激发学生的热情和兴趣是有帮助的。或许本书更适合那些成功地学习过微积分和线性代数初等课程的读者。这里对微积分的使用很少，而涉及的线性代数也不多，因此对于不熟悉它的读者阅读本书也不应该有任何问题。

自从第1版发行以来已经25年了，但本书仍然受到专业数学同行的欢迎，我很知足。

非常感谢鼓励我修订第4版以及向我提供有益建议的许多专家：Russ Rowlett（坎裴尔北加州大学），James Sellers（Penn州立大学），Michael Buchner（新墨西哥大学）。正如第3版一样，我特别感谢Leroy F.Meyers（俄亥俄州立大学）和Tom Zaslavsky（SUNY系统Binghamton分校），他们每位都向我提供了对第3版的广泛而详尽的建议。对于第4版，我得到了Nils Andersen（哥本哈根大学）、James Propp（威斯康星大学）和Louis Deatt（威斯康星大学）许多有用的建议，他们阅读了新版并对格路径的新的一节发表了看法。我希望本书继续反映我对组合数学的热爱以及我讲授这门课程的热情和讲授的方式。

最后，再次感谢我亲爱的妻子Mona，是她给我的生活带来幸福、激情和勇气。

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Contents

Chapter 1

What Is Combinatorics?

It would be surprising indeed if a reader of this book had never solved a combinatorial problem. Have you ever counted the number of games n teams would play if each team played every other team exactly once? Have you ever constructed magic squares? Have you ever attempted to trace through a network without removing your pencil from the paper and without tracing any part of the network more than once? Have you ever counted the number of poker hands that are full houses in order to determine what the odds against a full house are? These are all combinatorial problems. As they might suggest, combinatorics has its historical roots in mathematical recreations and games. Many problems that were studied in the past, either for amusement or for their aesthetic appeal, are today of great importance in pure and applied science. Today, combinatorics is an important branch of mathematics, and its influence continues to expand. Part of the reason for the tremendous growth of combinatorics has been the major impact that computers have had and continue to have in our society. Because of their increasing speed, computers have been able to solve large-scale problems that previously would not have been possible. But computers do not function independently. They need to be programmed to perform. The bases for these programs often are combinatorial algorithms for the solutions of problems. Analysis of these algorithms for efficiency with regard to running time and storage requirements requires more combinatorial thinking.

Another reason for the continued growth of combinatorics is its applicability to disciplines that previously had little serious contact with mathematics. Thus, we find that the ideas and techniques of combinatorics are being used not only in the traditional area of mathematical application, namely the physical sciences, but also in the social sciences, the biological sciences, information theory, and so on. In addition, combinatorics and combinatorial thinking have become more and more important in many mathematical disciplines.

Combinatorics is concerned with arrangements of the objects of a set into patterns satisfying specified rules. Two general types of problems occur repeatedly:

- *Existence of the arrangement.* If one wants to arrange the objects of a set so that certain conditions are fulfilled, it may not be at all obvious whether such an arrangement is possible. This is the most basic of questions. If the arrangement is not always possible, it is then appropriate to ask under what conditions, both necessary and sufficient, the desired arrangement can be achieved.

- *Enumeration or classification of the arrangements.* If a specified arrangement is possible, there may be several ways of achieving it. If so, one may want to count their number or to classify them into types.

Although both existence and enumeration can be considered for any combinatorial problem, it often happens in practice that, if the existence question requires extensive study, the enumeration problem is very difficult. However, if the existence of a specified arrangement can be settled with reasonable ease, it may be possible to count the number of ways of achieving the arrangement. In exceptional cases (when their number is small), the arrangements can be listed. It is important to understand the distinction between listing all the arrangements and determining their number. Once the arrangements are listed, they can be counted by setting up a one-to-one correspondence between them and the set of integers $\{1, 2, 3, \ldots, n\}$ for some n. This is the way we count: one, two, three, ... However, we shall be concerned primarily with techniques for determining the number of arrangements of a particular type without first listing them. Of course the number of arrangements may be so large as to preclude listing them all. In sum, many combinatorial problems are of the following forms:

"Is it possible to arrange . . . ?"

"Does there exist a . . . ?"

"In how many ways can . . . ?"

"Count the number of . . . ."

Two other combinatorial problems that occur in conjunction with these forms are the following:

- *Study of a known arrangement.* After one has done the (possibly difficult) work of constructing an arrangement satisfying certain specified conditions, its properties and structure can then be investigated. Such structure may have implications for the classification problem and also for potential applications. It may also have implications for the next problem.

- *Construction of an optimal arrangement.* If more than one arrangement is possible, one may want to determine an arrangement that satisfies some optimality criterion—that is, to find a "best" or "optimal" arrangement in some prescribed sense.

Thus, a general description of combinatorics might be that *combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures.* In this book, discrete generally means finite, although some discrete structures are infinite.

One of the principal tools of combinatorics for verifying discoveries is *mathematical induction.* Induction is a powerful procedure, and it is especially so in combinatorics. It is often easier to prove a stronger result than a weaker result with mathematical induction. Although it is necessary to verify more in the inductive step, the inductive hypothesis is stronger. Part of the art of mathematical induction is to find the right *balance* of hypotheses to carry out the induction. We assume that the reader is familiar with induction; he or she will become more so as a result of working through this book.

The solutions of combinatorial problems often require *ad hoc* arguments sometimes coupled with use of general theory. One cannot always fall back onto application of formulas or known results. One must set up a mathematical model, study the model, do some computation for small cases, develop some insight, and use one's own ingenuity for the solution of the problem. I do not mean to imply that there are no general principles or methods that can be applied. For counting problems, the inclusion–exclusion principle, the so-called pigeonhole principle, the methods of recurrence relations and generating functions, Burnside's theorem, and Pólya counting are all examples of general principles and methods that we will consider in later chapters. But, often, to see that they can be applied and how to apply them requires cleverness. Thus, experience in solving combinatorial problems is very important. *The implication is that with combinatorics, as with mathematics in general, the more problems one solves, the more likely one is able to solve the next problem.*

In order to make the preceding discussion more meaningful, let us now turn to a few examples of combinatorial problems. They vary from relatively simple problems (but requiring ingenuity for solution) to problems whose solutions were a major achievement in combinatorics. Some of these problems will be considered in more detail in subsequent chapters.

1.1 Example: Perfect Covers of Chessboards

Consider an ordinary chessboard which is divided into 64 squares in 8 rows and 8 columns. Suppose there is available a supply of identically shaped dominoes, pieces which cover exactly two adjacent squares of the chessboard. Is it possible to arrange 32 dominoes on the chessboard so that no 2 dominoes overlap, every domino covers 2 squares, and all the squares of the chessboard are covered? We call such an arrangement a *perfect cover* of the chessboard by dominoes. This is an easy arrangement problem, and one quickly can construct many different perfect covers. It is difficult but nonetheless possible to count the number of different perfect covers. This number was found by Fischer[1] in 1961 to be $12,988,816 = 2^4 \times (901)^2$. The ordinary chessboard can be replaced by a more general chessboard divided into mn squares lying in m rows and n columns. A perfect cover need not exist now. Indeed, there is no perfect cover for the 3-by-3 board. For which values of m and n does the m-by-n chessboard have a perfect cover? It is not difficult to see that an m-by-n chessboard will have a perfect cover if and only if at least one of m and n is even or, equivalently, if and only if the number of squares of the chessboard is even. Fischer has derived general formulae involving trigonometric functions for the number of different perfect covers for the m-by-n chessboard. This problem is equivalent to a famous problem in molecular physics known as the *dimer problem.* It originated in the investigation of the absorption of diatomic atoms (dimers) on surfaces. The squares of the chessboard correspond to molecules, while the dominoes correspond to the dimers.

Consider once again the 8-by-8 chessboard and, with a pair of scissors, cut out two diagonally opposite corner squares, leaving a total of 62 squares. Is it possible to arrange 31 dominoes to obtain a perfect cover of this "pruned" board? Although the pruned board is very close to being the 8-by-8 chessboard, which has over twelve million perfect covers, it has no perfect cover. The proof of this is an example of simple but clever combinatorial reasoning. In an ordinary 8-by-8 chessboard the squares are alternately colored black and white, with

[1] M.E. Fischer: Statistical Mechanics of Dimers on a Plane Lattice, *Physical Review*, 124 (1961), 1664-1672.

32 of the squares white and 32 of the squares black. If we cut out two diagonally opposite corner squares, we have removed two squares of the same color, say white. This leaves 32 black and 30 white squares. But each domino covers one black and one white square, so that 31 nonoverlapping dominoes on the board cover 31 black and 31 white squares. Therefore the pruned board has no perfect cover, and the reasoning above can be summarized by

$$31\boxed{B}\boxed{W} \neq 32\boxed{B} + 30\boxed{W}.$$

More generally, one can take an m-by-n chessboard whose squares are alternately colored black and white and arbitrarily cut out some squares, leaving a pruned board. When does a pruned board have a perfect cover? For a perfect cover to exist the pruned board must have an equal number of black and white squares. But this is not sufficient, as the example in Figure 1.1 indicates.

W	×	W	B	W
×	W	B	×	B
W	B	×	B	W
B	W	B	W	B

Figure 1.1

Thus, we ask: What are necessary and sufficient conditions for a pruned board to have a perfect cover? We will return to this problem in Chapter 9 and will obtain a complete solution by applying the theory of matchings in bipartite graphs. There, a practical formulation of this problem is given in terms of assigning applicants to jobs for which they qualify.

There is another way to generalize the problem of a perfect cover of an m-by-n board by dominoes. Let b be a positive integer. In place of dominoes we consider 1-by-b pieces that consist of b 1-by-1 squares joined side by side consecutively. We call these pieces *b-ominoes.* Thus, a b-omino can cover b consecutive squares in a row or b consecutive squares in a column. In Figure 1.2, a 5-omino is illustrated. A 2-omino is simply a domino. A 1-omino is called a *monomino.*

Figure 1.2. A 5-omino

A *perfect cover* of an m-by-n board by b-ominoes is an arrangement of b-ominoes on the board so that (i) no two b-ominoes overlap, (ii) every b-omino covers b squares of the board, and (iii) all the squares of the board are covered. *When does an m-by-n board have a perfect cover by b-ominoes?* Since each square of the board is covered by exactly one b-omino, in order for there to be a perfect cover b must be a factor of mn. Surely, a sufficient condition for the existence of a perfect cover is that b be a factor of m or b be a factor of n. For if b is a factor of m, we may perfectly cover the m-by-n board by arranging m/b b-ominoes in each of the n columns, while if b is a factor of n we may perfectly cover the board by arranging n/b b-ominoes in each of the m rows. Is this sufficient condition also necessary for there to be a perfect cover? Suppose for the moment that b is a prime number and that there is a perfect cover of the m-by-n board by b-ominoes. Then b is a factor of mn and, by a fundamental property of prime numbers, b is a factor of m or b is a factor of n. We conclude that, at least for the case of a prime number b, an m-by-n board can be perfectly covered by b-ominoes if and only if b is a factor of m or b is a factor of n.

In case b is not a prime number, we have to argue differently. So suppose we have the m-by-n board perfectly covered with b-ominoes. We want to show that either m or n has a remainder of 0 when divided by b. We divide m and n by b obtaining quotients p and q and remainders r and s, respectively:

$$\begin{aligned} m &= pb + r, \quad \text{where} \quad 0 \le r \le b-1, \\ n &= qb + s, \quad \text{where} \quad 0 \le s \le b-1. \end{aligned}$$

If $r = 0$, then b is a factor of m. If $s = 0$, then b is a factor of n. By interchanging the two dimensions of the board, if necessary, we may assume that $r \le s$. We then want to show that $r = 0$.

1	2	3	$\cdots$	$b-1$	b
b	1	2	$\cdots$	$b-2$	$b-1$
$b-1$	b	1	$\cdots$	$b-3$	$b-2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
2	3	4	$\cdots$	b	1

Figure 1.3. Coloring of a b-by-b board with b colors

We now generalize the alternate black-white coloring used in the case of dominoes ($b = 2$) to b colors. We choose b colors which we label as 1, 2, ... , b. We color a b-by-b board in the manner indicated in Figure 1.3, and we extend this coloring to an m-by-n board in the manner illustrated in Figure 1.4 for the case $m = 10$, $n = 11$, and $b = 4$.

Each b-omino of the perfect covering covers one square of each of the b colors. It follows that there must be the same number of squares of each color on the board. We consider the board to be divided into three parts: the upper pb-by-n part, the lower left r-by-qb part, and the lower right r-by-s part. (For the 10-by-11 board in Figure 1.4, we would have the upper 8-by-11 part, the 2-by-8 part in the lower left, and the 2-by-3 part in the lower right.) In the upper part each color occurs p times in each column and hence pn times altogether. In the lower left part each color occurs q times in each row and hence rq times altogether. Since each color occurs the same number of times on the whole board, it now follows that each color occurs the same number of times in the lower right r-by-s part.

1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2
3	4	1	2	3	4	1	2	3	4	1
2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4	1	2	3
4	1	2	3	4	1	2	3	4	1	2

Figure 1.4. Coloring of a 10-by-11 board with four colors

How many times does color 1 (and, hence, each color) occur in the r-by-s part? Since $r \leq s$, the nature of the coloring is such that color 1 occurs once in each row of the r-by-s part and hence r times in the r-by-s part. Let us now count the number of squares in the r-by-s part. On the one hand there are rs squares; on the other hand, there are r squares of each of the b colors and so rb squares altogether. Equating we get $rs = rb$. If $r \neq 0$, we cancel to get $s = b$, contradicting $s \leq b-1$. So $r = 0$, as desired. We summarize as follows:

An m-by-n board has a perfect cover by b-ominoes if and only if b is a factor of m or b is a factor of n.

A striking reformulation of the preceding statement is the following: Call a perfect cover *trivial* if all the b-ominoes are horizontal or all the b-ominoes are vertical. Then *an m-by-n board has a perfect cover by b-ominoes if and only if it has a trivial perfect cover.* Note that this does not mean that the only perfect covers are the trivial ones. It does mean that if a perfect cover is possible, then a trivial perfect cover is also possible.

1.2 Example: Cutting a Cube

Consider a block of wood in the shape of a cube, 3 feet on an edge. It is desired to cut the cube into 27 smaller cubes, 1 foot on an edge. What is the smallest number of cuts in which this can be accomplished? One way of cutting the cube is to make a series of 6 cuts, 2 in each direction, while keeping the cube in one block as shown in Figure 1.5. But is it possible to use fewer cuts if the pieces can be rearranged between cuts? An example is also given in Figure 1.5 where the second cut now cuts through more wood than it would have if we had not rearranged the pieces after the first cut. Since the number of pieces, and thus the number of rearrangements, increases with each cut, this might appear to be a difficult problem to analyze.

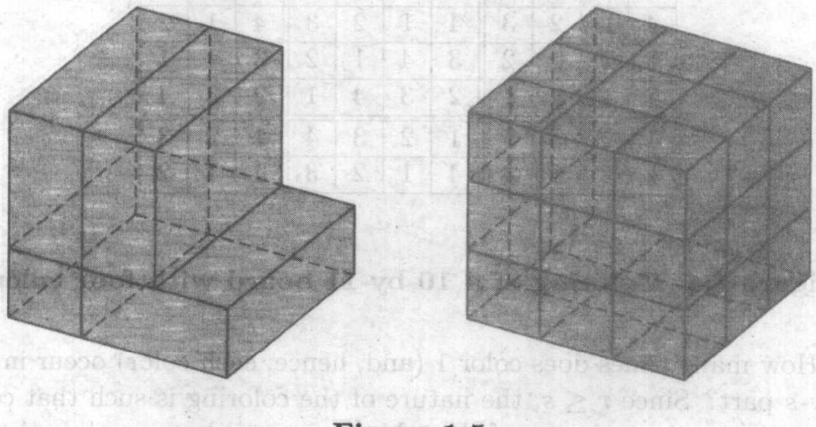

Figure 1.5

But let us look at it another way. Every one of the 27 small cubes except the one in the middle has at least one face that was originally part of one of the faces of the large cube. The cube in the middle has every one of its faces formed by cuts. Since it has 6 faces, 6 cuts are necessary to form it. Thus, at least 6 cuts are always necessary, and