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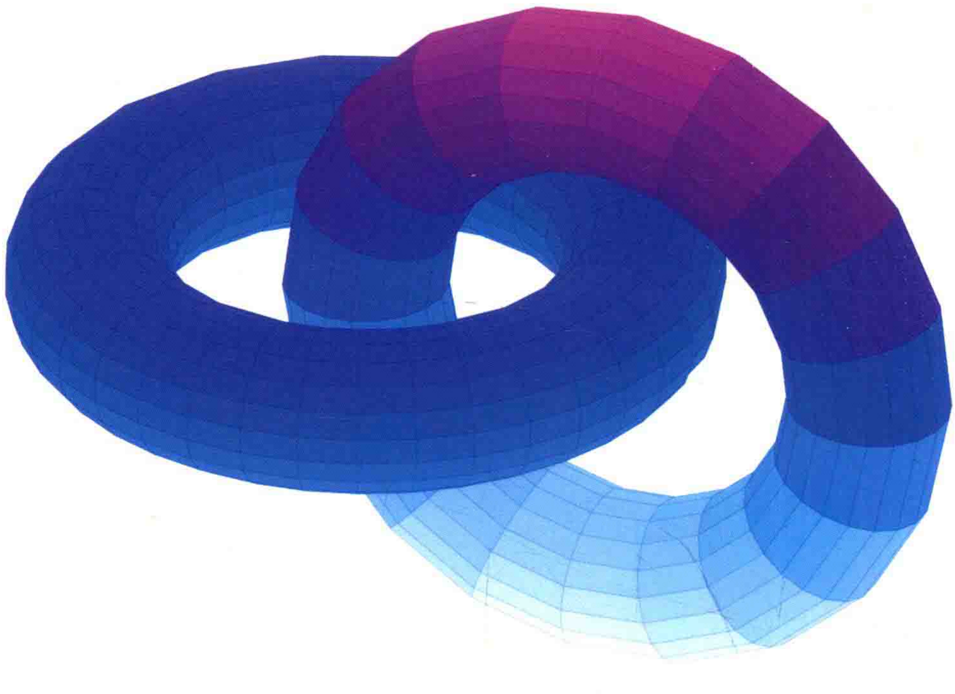
Gerardo Chacón, Humberto Rafeiro, Juan Camilo Vallejo

FUNCTIONAL ANALYSIS

A TERSE INTRODUCTION



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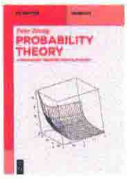

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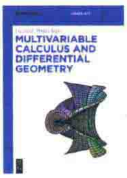

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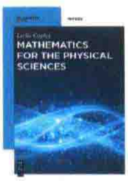

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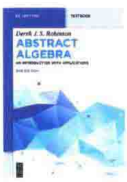

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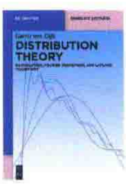

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A Yinzú, gracias por convivir con mis desvaríos
G.C.

à Daniela pelo seu amor e paciência
H.R.

Preface

This book is intended to make a smooth transition from linear algebra to the basics of functional analysis. When possible, we present concepts starting with a finite-dimensional example using matrix calculations and finite basis and we build up from that point to more complicated examples in infinite-dimensional spaces, making emphasis in the main differences from one case to the other. Although we rely upon reader's knowledge of basic linear algebra and vector spaces, we go back to indicate some relevant concepts and techniques that will be necessary in several parts in the book.

Throughout the book, there are several examples and proofs that are left to the reader to complete. This is done with the objective of requiring an active participation from the reader that we expect will result in a better insight of the problems and techniques involved in the theory.

The main prerequisite for this book would be a proof-based course in linear algebra. A basic course in topology is preferred but not completely necessary. Although measure-theoretic examples are presented in several parts of the book, those can be safely disregarded in a first reading. However, we would like to recommend Refs [8, 18].

In Chapter 1 we give a brief introduction to the axiom of choice and its equivalent formulations, emphasizing the usage of the principle. We give some examples which highlight the rationale in the handling of the axiom of choice. Among those examples are the proofs that every vector space has a Hamel basis, the existence of nonmeasurable Lebesgue sets and a whimsy version of the Banach–Tarski paradox.

Chapter 2 consists of an introduction to the theory of Hilbert spaces. It starts by defining the notion of *norm* as a way of measuring vectors and, consequently, distances between vectors. Then ℓ^p spaces are introduced and Minkowski and Hölder's inequalities are proven in order to define the $\|\cdot\|_p$-norm. We then notice that in the finite-dimensional context, the case $p = 2$ coincides with the Euclidean geometric manner of measuring vectors which comes from Pythagoras Theorem. We then exhibit the necessity of having a way of measuring angles in a vector space, and inner products are introduced along with several examples and geometric properties such as parallelogram identity and polarization identities. We finish the chapter studying orthogonal sequences and introducing the notion of Fourier coefficients and their properties.

In Chapter 3 we start by presenting examples of Banach spaces and show that in finite dimensions all norms are equivalent. We also show that the compactness of the closed unit ball is a property that characterizes finite-dimensional vector spaces. We describe some separable spaces and its relation to Schauder basis.

Chapter 4 contains the definition and examples of linear transformations in several vector spaces. Then we go back to the study of matrices as linear transformations acting on finite-dimensional spaces. We show that in this case, continuity is an automatic property and that this is a main difference from the infinite-dimensional case. We calculate the norm of some concrete bounded linear operators and show that the space of linear operators with range on a Banach space is itself a Banach space. Then we finish the chapter with an extension theorem.

After studying general linear operators, Chapter 5 is devoted to study the special case of linear functionals. First, by showing examples and then, in the case of Hilbert spaces, by identifying a way to represent all linear functionals. Next, we look for a similar representation in Banach spaces, motivating the notion of dual spaces and considering examples. The chapter finishes with a section presenting the bra-ket notation as a way of distinguishing between vectors in a Hilbert space and its dual.

Chapter 6 is an optional chapter focused on the theory of Fourier coefficients of functions on the space $L^2[-\pi, \pi]$. It starts by studying such coefficients from the point of view of a Hilbert space and then passes to the more general space $L^1[-\pi, \pi]$ in which the question of convergence of the Fourier series arises. Sufficient conditions on a function f are given to assure the pointwise and the uniform convergence of its Fourier series. This chapter uses tools from measure theory and topology of metric spaces and it can be disregarded if the reader does not have the necessary background.

Chapter 7 is also an optional chapter and relies heavily on measure theory and integration. We introduce the notion of convolution operator and obtain some immediate properties of this operator. The Young inequality for the convolution is also given. The notions of Dirac sequences and Friedrich mollifiers are introduced and it is shown that the convolution with a Friedrich mollifier generates an identity approximation operator in the framework of Lebesgue spaces. The Fourier transform is introduced in the case of L^1 functions and then, using the Plancherel Theorem, we extend the notion of Fourier transform into the space of square summable functions. Many properties of the Fourier transform are derived, e.g., the translation, modulation, convolution, uncertainty principle, among others. We end the chapter with a brief introduction to the Schwartz class of functions, which is the natural environment for the Fourier transform.

In Chapter 8 we study the Fixed Point Theorem in the realm of metric spaces. We give classical applications, e.g., the Babylonian algorithm, Newton's method, applications in the framework of differential and integral equations, to name a few. We also included a section touching on the subject of fractals, where we prove the Hutchinson Theorem.

Chapter 9 is devoted to the Baire Category Theorem. Although this is a theorem that belongs to topology it has plenty of applications in many branches of mathematics, since it is an existence theorem. We give the classical proof of Weierstrass of a continuous nowhere differentiable function and then we give a very short proof of the existence of such functions using Baire Category Theorem, in fact showing that the set of such functions is generic.

Chapters 10,11,12 and 13 are very important, since they contain the four pillars of functional analysis: the uniform boundedness principle, the Open Mapping Theorem, the Closed Graph Theorem and the Hahn–Banach Extension Theorem. All of these results are existence results and have many applications in analysis and elsewhere. In Chapter 10 we give two different proofs of the uniform boundedness principle, one of which does not rely on the Baire Category Theorem. As a particular case we show the Banach–Steinhaus Theorem. We introduce the notion of bilinear operator and show that a bilinear operator which is continuous in each coordinate is continuous. In Chapter 11 we define the notion of an open mapping and show the Open Mapping Theorem which states that a surjective bounded linear operator between Banach spaces is open. The so-called Banach Isomorphism Theorem is also derived. In Chapter 12 we study the notion of a closed operator and show the Closed Graph Theorem, which loosely states that a closed operator is continuous under certain conditions. Chapter 13 is devoted to the study of Hahn–Banach-type theorem. We give the real and complex extension version of the Hahn–Banach Theorem based on the Banach function and show the classical corollaries of this fact, e.g., the Hahn–Banach extension in norm form, the construction of a linear continuous functional with prescribed conditions, to name a few. The Minkowski functional is studied with some details and it is pointed out that in any vector space we can introduce a seminorm. Using the notion of Minkowski functional we study the separation theorems. We end the chapter with some applications of the Hahn–Banach Theorem, e.g., the result that all separable spaces are isometric isomorphic to a subspace of ℓ_∞, and with the Lax–Milgram Theorem.

In Chapter 14, we consider the concept of adjoint operators, starting again from the knowledge of matrices and motivating the definition with the relation between a real matrix and its transpose. The rigorous definition for Hilbert spaces is then presented and some properties are considered. In the case of Banach spaces, a new definition is given based on the modifications that are needed from the Hilbert space case. Properties and examples are also showed.

The weak topology is introduced in Chapter 15 as the smallest topology that makes all linear functionals continuous. Comparisons between weak and norm topologies are presented and the weak* topology is also presented. The chapter finishes with a section devoted to reflexive spaces, showing the classical example of ℓ^p spaces and the counter-example of the space ℓ^1.

Chapter 16 is dedicated to describe compact, normal and self-adjoint operators acting on Hilbert spaces, showing examples and properties in preparation to Chapter 17. There, a brief introduction to spectral theory is depicted, starting from a review of the situation in finite-dimensional complex spaces where a linear transformation can be represented by a diagonal matrix if, and only if, it is normal. This serves as a motivation to look for a similar result in general Hilbert spaces. The chapter ends by showing the spectral theorem for compact self-adjoint operators. In Chapter 18 we study in some detail the notion of compactness in metric spaces, we give several related notions and we provide some criteria of compactness for some function spaces.

In the preparation of this book we made extensive use of some works, namely Refs [20, 26, 24, 23, 30, 5, 21]. Many of the results, examples and proofs in the text are also taken from our personal notebooks, and the exact references were lost.

Washington, USA
Bogotá, Colombia

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H. Rafeiro and J.C. Vallejo
December 2016

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Basic Notation

Here we will review the basic notation that will be used throughout the book. In general, we will refer to a field $\mathbb{F}$ to denote either $\mathbb{R}$, the set of all real numbers, or $\mathbb{C}$, the set of complex numbers. Given a natural number n, $\mathbb{F}^n$ denotes the vector space of all n-tuples $(\alpha_1, \ldots, \alpha_n)$, where the α_j belong to either $\mathbb{R}$ or $\mathbb{C}$. The symbol $\left[\alpha_{ij}\right]_{i,j=1}^{n}$ denotes an $n \times n$ matrix with entries α_{ij}. The set of natural numbers will be denoted as $\mathbb{N}$.

The absolute value of a real number α will be denoted by $|\alpha|$. Similarly, if z is a complex number, then we will use the symbol $|z|$ to denote the absolute value or modulus of a complex number. If $z = a + ib$, then the complex conjugated will be denoted as $\overline{z} = a - ib$.

Given a topological space X and a set $E \subseteq X$, we denote by $\overline{E}$ the closure of the set E. If there is a metric d defined on X, then $B_r(x)$ will denote the open ball centered at x and with radius r. Similarly, $\overline{B}_r(x)$ denotes the closed ball centered at x and with radius r. Sequences of elements will be denoted as $(x_n)_{n=1}^{\infty}$, and with the symbol $x_n \to x$ we represent that the limit of x_n is equal to x when n tends to infinity. The topology in which such limits are considered should be clear from the context.

The symbols $\sup(A)$ and $\inf(A)$ will denote, respectively, the supremum and the infimum of a set $A \subseteq \mathbb{R}$.

We use the symbol $\oslash$ to indicate the end of an example or a remark. Similarly, the symbol $\square$ will denote the end of a proof.

Finally, references to the books in the bibliography will be denoted by square brackets.

Contents

1 Choice Principles

Learning Targets

- Introduction to the axiom of choice.
- Learn some *choice principles* which are equivalent to the axiom of choice.
- Get acquainted with and understand how to use some type of choice principles in a *real-world situation*.

1.1 Axiom of Choice

The axiom of choice is a devise used when we need to iterate some process infinitely, e.g., when we need to choose some infinite elements from a set. One of the formulations of the axiom uses the notion of *choice function* and the others rely on the *Cartesian product*.

Definition 1.1 (Choice Function). Let X be a nonempty set. A function $f : 2^X \longrightarrow X$ is said to be a choice function on the set X if $f(A) \in A$ when $\emptyset \neq A \subseteq X$.

With the notion of choice function we can state the axiom of choice.

Axiom 1.2 (Axiom of Choice). *For every nonempty set there exists a choice function.*

The axiom of choice can be phrased as

> For every family A of disjoint nonempty sets there exists a set B which has exactly one element in common with each set belonging to A.

Although this axiom seems to be true, at least for the case of finite choices or numerable choices, in the case of nonnumerable infinite choices, the situation is quite different. Even for the numerable choices it is somewhat an illusion, since we do not have a way to guarantee that we can choose all the numerable elements without resorting to some type of axiom of choice. For example, to guarantee the existence of the natural numbers $\mathbb{N}$ it is necessary to assume some type of axiom of infinity, as is done in the Zermelo–Fraenkel set theory. The power of the axiom of choice lies in the fact that it permits to choose an *infinite* number of things *instantaneously*. In Ref. [4] we have the following remark:

> Several mathematicians claimed that proofs involving the axiom of choice have a different nature from proofs not involving it, because the axiom of choice is a unique set theoretical principle

which states the existence of a set without giving a method of defining ("constructing") it, i.e. is not effective.

In the beginning of the twentieth century there were disputes among several renowned mathematicians regarding the acceptance of this axiom. One of the serious obstacles is the so-called *Banach–Tarski paradox.*

Banach–Tarski Paradox: The unit ball $B := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ in three dimensions can be disassembled into a finite number of pieces (in fact, just five pieces would suffice), which can then be reassembled (after translating and rotating each of the pieces) to form two disjoint copies of the ball B.

For more information regarding the Banach–Tarski paradox, see Ref. [41]. In Theorem 1.6 we give a whimsy version of the Banach–Tarski paradox.

Nowadays the axiom of choice is accepted by the majority of the mathematical community and we will add it to our mathematical toolbox without further philosophical discourse. For an account of the history of the axiom of choice see Ref. [29].

From now on in the whole book, when we use the tribar symbol ⚠ before a result, it means that it relies on the axiom of choice or any of its equivalent formulations.

Definition 1.3. Let $(X_j)_{j\in J}$ be a family of sets. The *Cartesian product*, denoted by

$$\Pi_{j\in J} X_j,$$

is the set of all maps $x : j \to \cup_{j\in} X_j$ such that $x(j) \in X_j$ for any $j \in J$.

In the next theorem we collect several choice principles that are equivalent to the axiom of choice, but before doing that we will need a couple of definitions.

Definition 1.4. Let $(X, \preccurlyeq)$ be a poset (partially ordered set). We say that a subset S, with $\emptyset \neq S \subseteq X$, is a *chain in* X if all the elements in S are related by the partial order from X, i.e. if for all $x, y \in S$ we have either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 1.5. A poset X is *well ordered* if any nonempty subset of X has a minimum element, viz. if $\min(A)$ exists whenever $\emptyset \neq A \subseteq X$.

The following theorem will be given without proof; for the proof and further results, cf. Refs [9, 16, 17].

Theorem 1.6. *The following principles are equivalent:*