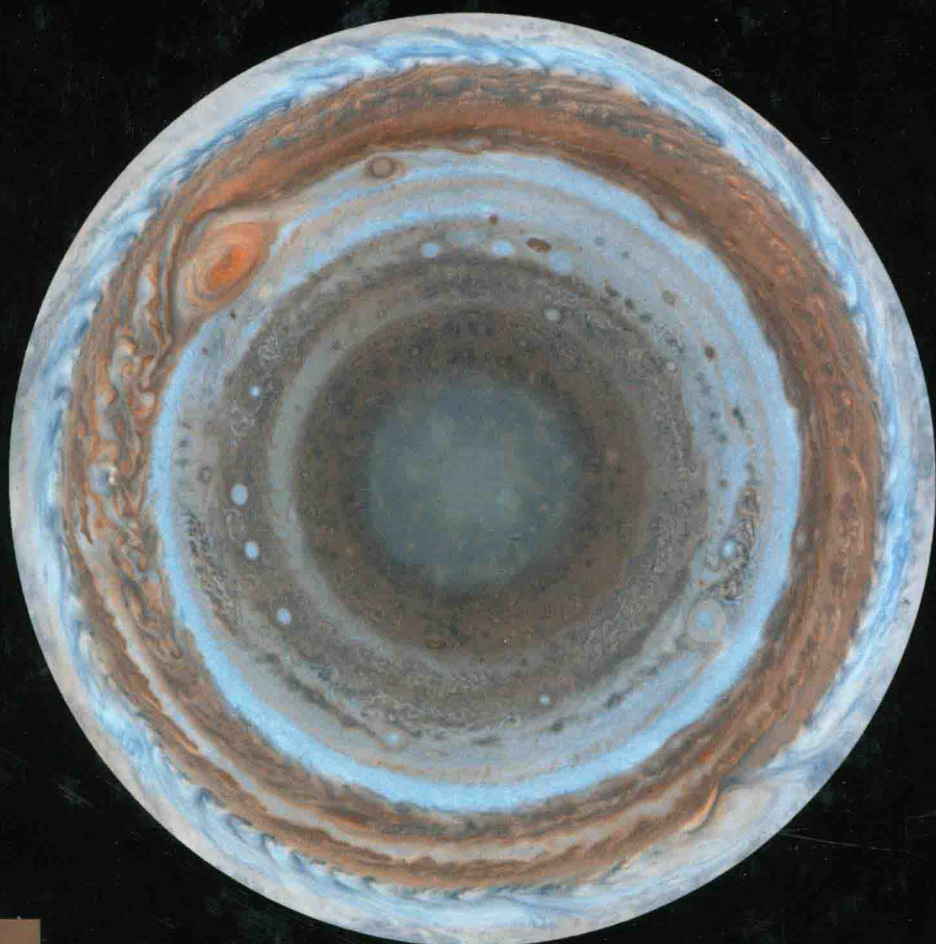


Turbulence in Rotating, Stratified and Electrically Conducting Fluids

P. A. DAVIDSON



CAMBRIDGE

TURBULENCE IN ROTATING, STRATIFIED AND ELECTRICALLY CONDUCTING FLUIDS

P. A. DAVIDSON

University of Cambridge



CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107026865

© P. A. Davidson 2013

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2013

Printed in Spain by Grafos SA, Arte sobre papel

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-02686-5 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Every effort has been made to secure necessary permissions to reproduce copyright material in this work, though in some cases it has proved impossible to trace copyright holders. If any omissions are brought to our notice, we will be happy to include appropriate acknowledgements on reprinting.

TURBULENCE IN ROTATING, STRATIFIED AND ELECTRICALLY CONDUCTING FLUIDS

There are two recurring themes in astrophysical and geophysical fluid mechanics: waves and turbulence. This book investigates how turbulence responds to rotation, stratification or magnetic fields, identifying common themes, where they exist, as well as the essential differences which inevitably arise between different classes of flow.

The discussion is developed from first principles, making the book suitable for graduate students as well as professional researchers. The author focusses first on the fundamentals and then progresses to such topics as the atmospheric boundary layer, turbulence in the upper atmosphere, turbulence in the core of the Earth, zonal winds in the giant planets, turbulence within the interior of the Sun, the solar wind, and turbulent flows in accretion discs. The book will appeal to engineers, geophysicists, astrophysicists and applied mathematicians who are interested in naturally occurring turbulent flows.

P. A. DAVIDSON is Professor of Fluid Mechanics in the Department of Engineering at the University of Cambridge.

For Sarah Elizabeth and James Alexander

Preface

I love deadlines. I especially love the swooshing sound they make as they go flying by.

Douglas Adams

This is a book I had in mind to write for some years, but self-imposed deadlines came and went to little avail. It was not until late 2008, inspired by an Isaac Newton Institute programme on turbulence, that lethargy finally gave way to action.

There are two recurring themes in astrophysical and geophysical fluid mechanics: waves and turbulence. These flows are generally subject to a background rotation, strong stratification, or an ambient magnetic field, and in all three cases this allows the fluid to sustain internal wave motion. Moreover, such flows are almost invariably turbulent, and the turbulence is often central to their behaviour, allowing accretion discs to feed mass to young or dying stars, triggering explosions on the surface of the Sun, diffusing heat, momentum and pollutants across the atmospheric boundary layer, and generating the terrestrial magnetic field deep within the interior of the Earth. Sometimes the waves and turbulence coexist with little interaction, but more commonly there is an interplay between the two. For example, in some flows the turbulence excites waves which, in turn, reshape the structure of the turbulence by dispersing the energy held in vortices. Conversely, at times internal waves grow and become unstable, initiating new turbulence. On yet other occasions the turbulence displays almost no wave-like properties, despite a background rotation or stratification. There appears to be a multitude of possibilities.

Understanding this two-way interaction between waves and turbulence, where and when it occurs, has proven to be a formidable challenge. When the turbulence is very weak (relative to the wave motion) there are well-established mathematical techniques that can be brought to bear on the problem, but unfortunately turbulence in nature is rarely weak, and so we have few mathematical formalisms at our disposal. As with conventional turbulence, much rests on dimensional analysis, heuristic physical arguments, and careful numerical or physical experiments. Moreover, the nature and extent of this wave–turbulence interaction varies markedly from case to case, being quite different for, say, internal gravity waves, inertial waves maintained by the Coriolis force, and Alfvén waves which travel along magnetic field lines. In the case of rapidly rotating turbulence, some progress has been

made and it is, perhaps, possible to rationalise the observed anisotropic structuring of the large eddies in terms of inertial wave propagation, though there are many details still to be resolved. And in magnetohydrodynamic turbulence the observed distribution of energy across the various scales can now be explained in terms of the interaction of Alfvén waves with turbulent eddies. In stratified turbulence, however, the significance of gravity waves, and the manner in which they interact with the turbulence, is still poorly understood, and indeed in some instances it is, perhaps, not terribly helpful to try and interpret events in terms of wave–turbulence interactions.

Any author embarking on a book on geophysical and astrophysical turbulence is immediately faced with a number of problems, not the least of which is that many of the central issues remain unresolved, or at least only partially understood. There is disagreement, for example, as to why rapidly rotating turbulence is dominated by cyclonic columnar vortices, or why strongly stratified turbulence takes the form of flat, pancake-like eddies (at least at the large scales). So this is a story without an ending. A second difficulty is that many diverse communities study such flows (meteorologists, oceanographers, astrophysicists . . .) and these communities have tended to develop their own language and ways of conceiving the phenomena. Communication between these groups is not always straightforward. Yet, despite all these difficulties, it seems natural to seek to provide an overview, if only a partial one, of these distinct yet closely related areas of study.

Given the difficulty of the subject matter, the open-ended nature of the problem (or rather problems), and the difficulties of language, prudence dictates that any text on the subject must have modest aims. Certainly this book makes no claims for completeness; indeed, entire books could be (have been) devoted to, say, turbulent motion in accretion discs, or in the Sun, or in the atmospheric boundary layer. Rather, our aim here is to take a step back and provide an account of how turbulence responds to rotation, stratification and magnetic fields, identifying common themes where they exist, as well as the essential differences which inevitably arise. In order to counter the issue of language, it was decided to develop the entire subject more or less from first principles, and so the book starts with extended chapters on the theory of rotating fluids, stratified flows, and magnetohydrodynamics, all in the absence of turbulence. This constitutes Part I of the text. Turbulence too tends to be shrouded in its own language and mysteries, and so turbulence theory is also introduced and developed from first principles (Part II of the book). It is not until we reach Part III of this text that we arrive at the core of the problem, where turbulence is combined with rotation, stratification and magnetic fields. Here we encounter the recurring difficulty that many of the central questions remain unanswered, and that often there are competing explanations for the observed phenomena. I have tried to pick my way carefully through this minefield, mentioning controversies where they exist, and avoiding topics and theories that seem likely to date rather quickly. While I hope the outcome is broadly satisfactory, I have lived long enough to be quite familiar with my own imperfections, and so I beg the reader to be indulgent if, at times, they find the balance is not to their taste.

It is a pleasure to acknowledge the help of friends and colleagues. Over the years I have benefited from many interesting discussions on turbulence with Julian Hunt, Yukio Kaneda,

Per-Åge Krogstad and Keith Moffatt. Kate Graham helped with some of the figures, Jim Riley introduced me to the mysteries of geophysical turbulence, Uli Christensen was kind enough to share his thoughts on recent geodynamo simulations, and Alex Schekochihin helped guide me through the labyrinth of spectral theories of MHD turbulence. David Tranah of Cambridge University Press was a delight to work with and helped shape this book. Finally, I have been blessed with a long-suffering wife who has patiently endured those unreasonably long Sunday silences which inevitably accompanied the writing of this book.

Contents

<i>Preface</i>	<i>page xv</i>
1 The interplay of waves and turbulence: a preview	1
1.1 Three types of wave	1
1.2 Waves and turbulence	11
1.3 Turbulence in geophysical and astrophysical flows	17
PART I FROM FLUID MECHANICS TO MAGNETOHYDRODYNAMICS	
2 Elementary fluid dynamics	27
2.1 The Navier–Stokes equation	27
2.2 The dissipation of energy in a viscous fluid	31
2.3 The vorticity equation	33
2.4 Burgers’ vortex	36
2.5 Kelvin’s theorem and Helmholtz’s laws	39
2.6 Conservation of helicity	41
2.7 The dynamics of a localised vorticity distribution	43
3 Motion in a rotating fluid	50
3.1 The Coriolis force	51
3.2 The Taylor–Proudman theorem	52
3.3 Inertial waves and the formation of Taylor columns	53
3.3.1 Inertial waves	53
3.3.2 The spontaneous growth of Taylor columns	57
3.3.3 The helical structure of inertial waves	61
3.4 Waves and stability	62
3.5 Rossby waves: an example of quasigeostrophic flow	64
3.6 Rotating, shallow-water flow	66
3.6.1 The shallow-water equations and potential vorticity conservation	66
3.6.2 Small disturbances: Poincaré waves and geostrophic adjustment	69
3.6.3 Small disturbances near a boundary: Kelvin waves	72

3.7	Quasigeostrophic, shallow-water flow	74
3.7.1	The quasigeostrophic shallow-water (QGSW) equations	74
3.7.2	The QGSW equations from potential vorticity conservation	75
3.7.3	The QGSW equations with bottom topography and the β -plane	77
3.7.4	Rossby waves revisited	78
3.8	The boundary layers of Karman, Bödewadt and Ekman	80
4	Motion in a stratified fluid	92
4.1	The Boussinesq approximation and the suppression of vertical motion	92
4.2	Blocking: the analogue of Taylor columns	97
4.3	Lee waves: the analogue of stationary inertial waves	101
4.4	Internal gravity waves	104
4.4.1	Linear gravity waves	104
4.4.2	Waves in rotating, stratified fluids	107
4.5	Potential vorticity revisited	107
4.6	Valley winds: the analogue of Ekman layers	112
5	The equations of electrodynamics	117
5.1	Maxwell's equations	117
5.2	Integral versions of Ampère's and Faraday's laws	120
5.3	An evolution equation for the magnetic field	123
5.4	The Lorentz force, Maxwell's stresses and Faraday's tension	124
5.5	The exchange of energy between the magnetic field and velocity field	126
6	Motion in a conducting fluid: magnetohydrodynamics	129
6.1	The equations of MHD and key dimensionless groups	129
6.2	Kinematics	132
6.2.1	Ideal fluids: Alfvén's theorem and magnetic helicity	132
6.2.2	Diffusive effects: stretched flux tubes and flux expulsion	135
6.3	Magnetic damping at low magnetic Reynolds number	138
6.3.1	Simplifications at low R_m	138
6.3.2	Energy destruction and momentum conservation	139
6.3.3	Damping of a vortex	140
6.4	The damping of turbulence at arbitrary R_m (a preview)	144
6.5	Dynamics at high magnetic Reynolds number	146
6.5.1	Alfvén waves and Elsässer variables	146
6.5.2	Magnetostrophic waves	148
6.5.3	Conservation of cross helicity	150
7	Instabilities and transition to turbulence	157
7.1	The instabilities of Rayleigh, Taylor and Bénard	157
7.1.1	Rayleigh's centrifugal instability and Taylor–Couette flow	157
7.1.2	Rayleigh–Bénard convection, with and without a magnetic field	161

7.2	Stability of a stratified shear flow	166
7.2.1	Shear flow in the absence of stratification: Rayleigh's inflection-point theorem	167
7.2.2	The Kelvin–Helmholtz instability	169
7.2.3	A necessary condition for instability of a stratified shear flow	173
7.3	Stability of MHD equilibria in ideal fluids	175
7.3.1	The stability of static equilibria	175
7.3.2	The stability of non-static equilibria	179
7.3.3	A Hamiltonian approach to stability: the role of the Lagrangian	182
7.3.4	An aside: the Kelvin–Arnold variational principle for Euler flows	185
7.3.5	The Chandrasekhar–Velikhov instability (or MRI) (i): a model problem	187
7.3.6	The MRI (ii): the case of rotation plus an azimuthal field	190
7.3.7	The MRI (iii): the case of rotation plus an axial field	191

PART II TURBULENCE IN THE ABSENCE OF BODY FORCES

8	Elementary properties of turbulence	197
8.1	Transition to turbulence: some common themes	198
8.2	The need for a statistical approach and the closure problem of turbulence	204
8.3	Different scales in a turbulent flow and the zeroth law of turbulence	206
8.4	Richardson's energy cascade and Kolmogorov's microscales	209
8.5	Enstrophy production in a turbulent flow	216
9	The language of turbulence: kinematics and statistics	220
9.1	Velocity correlation functions and structure functions	220
9.1.1	Correlation functions	221
9.1.2	Structure functions	227
9.1.3	Skewness and flatness	230
9.2	Fourier space	233
9.2.1	The Fourier transform as a filter	233
9.2.2	The spectral tensor and the energy spectrum	236
9.3	The simplifications of isotropy	243
9.3.1	Correlation functions and structure functions in isotropic turbulence	244
9.3.2	The spectral tensor in isotropic turbulence	247
9.3.3	Relating the second-order structure function to the energy spectrum	250
9.3.4	Isotropic turbulence with helicity	253
9.3.5	Axisymmetric turbulence	254

10 Hydrodynamic turbulence I: classical theories	259
10.1 The phenomenology of Richardson and Kolmogorov	259
10.1.1 Richardson's cascade (reprise)	259
10.1.2 Kolmogorov's theory of the small scales	266
10.1.3 The Kolmogorov–Obukhov–Corrsin model of passive scalar mixing	277
10.1.4 Yaglom's four-thirds law and Corrsin's integral in scalar mixing	281
10.2 Vortex and material line stretching	283
10.2.1 The enstrophy budget	283
10.2.2 Enstrophy production and the skewness factor: Betchov's theory	286
10.2.3 The stretching of material lines	290
10.2.4 Richardson's law of two-particle diffusion	293
10.3 The Karman–Howarth equation and its immediate consequences	297
10.3.1 The Karman–Howarth equation	297
10.3.2 Kolmogorov's four-fifths law	300
10.3.3 The skewness factor and enstrophy production (reprise)	301
10.3.4 Dynamics of the third-order correlations and the problem of closure	303
10.3.5 Dynamics in spectral space	305
10.4 Kolmogorov's refined model of the small scales	307
11 Hydrodynamic turbulence II: steps towards rotating, stratified and MHD turbulence	318
11.1 The evolution of the large scales	319
11.1.1 Isotropic turbulence: Saffman versus Batchelor turbulence	319
11.1.2 Long-range interactions in Saffman and Batchelor turbulence	325
11.1.3 The decay laws of Kolmogorov and Saffman for isotropic turbulence	328
11.1.4 Saffman's analysis of anisotropic turbulence	330
11.1.5 A proof of the invariance of the Saffman integrals L_{ij} in anisotropic turbulence	335
11.1.6 Axisymmetric Saffman turbulence	338
11.1.7 The role of angular momentum conservation in isotropic Batchelor turbulence: Landau's theory	343
11.1.8 Problems with Landau's theory of Batchelor turbulence	345
11.1.9 A consistent theory of the large scales in Batchelor turbulence	348
11.2 Two-dimensional turbulence	349
11.2.1 Vortex dynamics in two dimensions	350
11.2.2 The classical theory of Batchelor	355
11.2.3 The role of the coherent vortices	360
11.2.4 The governing equations in statistical form	362
11.2.5 Batchelor revisited	366
11.2.6 Statistical invariants associated with the large scales	368

PART III TURBULENCE IN THE PRESENCE OF BODY FORCES

12	Rapidly rotating turbulence	381
12.1	The early experimental observations	382
12.2	Structure formation through wave propagation	386
12.2.1	The shaping of a single eddy by linear inertial wave radiation	387
12.2.2	Implications for homogeneous turbulence: some more linear theory	391
12.2.3	Anisotropic structuring via non-linear wave interactions: resonant triads	394
12.3	Recent experimental evidence	398
12.4	The cyclone, anticyclone asymmetry	404
12.5	The rate of decay of energy	405
12.5.1	A Saffman-like invariant for rapidly rotating turbulence	406
12.5.2	Speculative decay laws	409
13	Towards geophysics: shallow-water, rapidly rotating turbulence	414
13.1	Governing equations	414
13.2	Statistical invariants	416
13.3	Turbulence on the β -plane: waves versus turbulence	419
13.4	Zonal flows in β -plane turbulence	423
13.5	Spectra in β -plane turbulence	428
14	Homogeneous stratified turbulence	435
14.1	Governing equations and dimensionless groups	435
14.2	Scalings, regimes and structures	439
14.3	A spectral description of stratified turbulence	444
14.4	The experimental and numerical evidence	446
14.5	Open questions and speculative spectral scalings	450
14.6	The rate of energy decay	453
14.6.1	A Saffman-like invariant	453
14.6.2	Possible decay laws	455
14.7	An alternative approach: the inhomogeneous turbulent cloud	457
15	Stratified shear flows and the atmospheric boundary layer	463
15.1	Neutral shear flows	463
15.1.1	The log-law of the wall for momentum and temperature	463
15.1.2	The k^{-1} and $\ln(r/z)$ laws	470
15.2	The equations of stratified shear flow and the flux Richardson number	479
15.3	The atmospheric boundary layer	484
15.3.1	The structure of the ABL and the diurnal cycle	485
15.3.2	Prandtl's theory for weak mean shear	487
15.3.3	The Monin–Obukhov theory	489

15.3.4 Spectral measurements: near-neutral and stable conditions	492
15.3.5 More spectral measurements: the unstable ABL	496
16 MHD turbulence at low magnetic Reynolds number	502
16.1 Governing equations	503
16.2 Angular momentum conservation, the growth of anisotropy and the decay of energy	504
16.3 The evolution of individual eddies	507
16.4 From angular momentum conservation to statistical invariants	513
16.5 A Loitsyansky-like invariant for homogeneous MHD turbulence	516
16.6 The numerical evidence for a Loitsyansky-like invariant	519
16.7 A Saffman-like invariant for homogeneous MHD turbulence	522
16.8 Possible decay laws for fully developed $E(k \rightarrow 0) \sim k^2$ and $E(k \rightarrow 0) \sim k^4$ turbulence	522
16.9 The numerical evidence for freely decaying $E(k \rightarrow 0) \sim k^4$ turbulence	525
17 Turbulence in the core of the Earth: the geodynamo	528
17.1 The need for a geodynamo theory	528
17.2 The structure of the Earth and the geomagnetic field	530
17.3 Some elementary ideas in dynamo theory	535
17.4 Anti-dynamo theories and necessary bounds	546
17.4.1 A minimum value of R_m is needed	546
17.4.2 Cowling's theorem and its relatives	547
17.5 Parker's model of the geodynamo	549
17.6 Two-scale theories of the geodynamo	553
17.7 The Taylor constraint	556
17.8 The numerical simulations	557
17.9 Other planetary dynamos	561
18 MHD turbulence at high magnetic Reynolds number	573
18.1 Two-dimensional MHD turbulence	573
18.1.1 Governing equations, ideal invariants and cascade directions	573
18.1.2 The evidence of the numerical simulations	578
18.2 Free decay and the Landau–Loitsyansky and Saffman invariants revisited	582
18.3 The spontaneous growth of a seed field and the importance of Pr_m	585
18.4 Magnetic field generation in non-helical forced turbulence	587
18.4.1 The magnetic microscale, η_λ , of a seed field	588
18.4.2 Kazantsev's kinematic model	589
18.4.3 Saturation of the magnetic field	593
18.5 Helical turbulence and selective decay	595
18.5.1 Ideal invariants revisited	595