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Thomas A. Garrity 著

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All the Mathematics You Missed

But Need to Know for Graduate School

清华大学出版社

Beginning graduate students in mathematics and other quantitative subjects are expected to have a daunting breadth of mathematical knowledge, but few have such a background. This book will help students see the broad outline of mathematics and to fill in the gaps in their knowledge.

The author explains the basic points and a few key results of the most important undergraduate topics in mathematics, emphasizing the intuitions behind the subject. The topics include linear algebra, vector calculus, differential geometry, real analysis, point-set topology, differential equations, probability theory, complex analysis, abstract algebra, and more. An annotated bibliography offers a guide to further reading and more rigorous foundations.

This book will be an essential resource for advanced undergraduate and beginning graduate students in mathematics, the physical sciences, engineering, computer science, statistics, and economics, and for anyone else who needs to quickly learn some serious mathematics.

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Thomas A. Garrity is Professor of Mathematics at Williams College in Williamstown, Massachusetts. He was an undergraduate at the University of Texas, Austin, and a graduate student at Brown University, receiving his Ph.D. in 1986. From 1986 to 1989, he was G.C. Evans Instructor at Rice University. In 1989, he moved to Williams College, where he has been ever since except in 1992–3, when he spent the year at the University of Washington, and 2000–1, when he spent the year at the University of Michigan, Ann Arbor.

Dedicated to the Memory

of

Robert Mizner

Preface

Math is Exciting. We are living in the greatest age of mathematics ever seen. In the 1930s, there were some people who feared that the rising abstractions of the early twentieth century would either lead to mathematicians working on sterile, silly intellectual exercises or to mathematics splitting into sharply distinct subdisciplines, similar to the way natural philosophy split into physics, chemistry, biology and geology. But the very opposite has happened. Since World War II, it has become increasingly clear that mathematics is one unified discipline. What were separate areas now feed off of each other. Learning and creating mathematics is indeed a worthwhile way to spend one's life.

Math is Hard. Unfortunately, people are just not that good at mathematics. While intensely enjoyable, it also requires hard work and self-discipline. I know of no serious mathematician who finds math easy. In fact, most, after a few beers, will confess as to how stupid and slow they are. This is one of the personal hurdles that a beginning graduate student must face, namely how to deal with the profundity of mathematics in stark comparison to our own shallow understandings of mathematics. This is in part why the attrition rate in graduate school is so high. At the best schools, with the most successful retention rates, usually only about half of the people who start eventually get their PhDs. Even schools that are in the top twenty have at times had eighty percent of their incoming graduate students not finish. This is in spite of the fact that most beginning graduate students are, in comparison to the general population, amazingly good at mathematics. Most have found that math is one area in which they could shine. Suddenly, in graduate school, they are surrounded by people who are just as good (and who seem even better). To make matters worse, mathematics is a meritocracy. The faculty will not go out of their way to make beginning students feel good (this is not the faculty's job; their job is to discover new mathematics). The fact is that there are easier (though, for a mathematician, less satisfying) ways to make a living. There is truth in the statement that you must be driven to become a mathematician.

Mathematics is exciting, though. The frustrations should more than be compensated for by the thrills of learning and eventually creating (or discovering) new mathematics. That is, after all, the main goal for attending graduate school, to become a research mathematician. As with all creative endeavors, there will be emotional highs and lows. Only jobs that are routine and boring will not have these peaks and valleys. Part of the difficulty of graduate school is learning how to deal with the low times.

Goal of Book. The goal of this book is to give people at least a rough idea of the many topics that beginning graduate students at the best graduate schools are assumed to know. Since there is unfortunately far more that is needed to be known for graduate school and for research than it is possible to learn in a mere four years of college, few beginning students know all of these topics, but hopefully all will know at least some. Different people will know different topics. This strongly suggests the advantage of working with others.

There is another goal. Many nonmathematicians suddenly find that they need to know some serious math. The prospect of struggling with a text will legitimately seem for them to be daunting. Each chapter of this book will provide for these folks a place where they can get a rough idea and outline of the topic they are interested in.

As for general hints for helping sort out some mathematical field, certainly one should always, when faced with a new definition, try to find a simple example and a simple non-example. A non-example, by the way, is an example that almost, but not quite, satisfies the definition. But beyond finding these examples, one should examine the reason why the basic definitions were given. This leads to a split into two streams of thought for how to do mathematics. One can start with reasonable, if not naive, definitions and then prove theorems about these definitions. Frequently the statements of the theorems are complicated, with many different cases and conditions, and the proofs are quite convoluted, full of special tricks.

The other, more mid-twentieth century approach, is to spend quite a bit of time on the basic definitions, with the goal of having the resulting theorems be clearly stated and having straightforward proofs. Under this philosophy, any time there is a trick in a proof, it means more work needs to be done on the definitions. It also means that the definitions themselves take work to understand, even at the level of figuring out why anyone would care. But now the theorems can be cleanly stated and proved.

In this approach the role of examples becomes key. Usually there are basic examples whose properties are already known. These examples will shape the abstract definitions and theorems. The definitions in fact are made in order for the resulting theorems to give, for the examples, the answers we expect. Only then can the theorems be applied to new examples and cases whose properties are unknown.

For example, the correct notion of a derivative and thus of the slope of a tangent line is somewhat complicated. But whatever definition is chosen, the slope of a horizontal line (and hence the derivative of a constant function) must be zero. If the definition of a derivative does not yield that a horizontal line has zero slope, it is the definition that must be viewed as wrong, not the intuition behind the example.

For another example, consider the definition of the curvature of a plane curve, which is in Chapter Seven. The formulas are somewhat ungainly. But whatever the definitions, they must yield that a straight line has zero curvature, that at every point of a circle the curvature is the same and that the curvature of a circle with small radius must be greater than the curvature of a circle with a larger radius (reflecting the fact that it is easier to balance on the earth than on a basketball). If a definition of curvature does not do this, we would reject the definitions, not the examples.

Thus it pays to know the key examples. When trying to undo the technical maze of a new subject, knowing these examples will not only help explain why the theorems and definitions are what they are but will even help in predicting what the theorems must be.

Of course this is vague and ignores the fact that first proofs are almost always ugly and full of tricks, with the true insight usually hidden. But in learning the basic material, look for the key idea, the key theorem and then see how these shape the definitions.

Caveats for Critics. This book is far from a rigorous treatment of any topic. There is a deliberate looseness in style and rigor. I am trying to get the point across and to write in the way that most mathematicians talk to each other. The level of rigor in this book would be totally inappropriate in a research paper.

Consider that there are three tasks for any intellectual discipline:

1. Coming up with new ideas.

2. Verifying new ideas.

3. Communicating new ideas.

How people come up with new ideas in mathematics (or in any other field) is overall a mystery. There are at best a few heuristics in mathematics, such as asking if something is unique or if it is canonical. It is in verifying new ideas that mathematicians are supreme. Our standard is that there must be a rigorous proof. Nothing else will do. This is why the mathematical literature is so trustworthy (not that mistakes don't creep in, but they are usually not major errors). In fact, I would go as far as to say that if any discipline has as its standard of verification rigorous proof, than that discipline must be a part of mathematics. Certainly the main goal for a math major in the first few years of college is to learn what a rigorous proof is.

Unfortunately, we do a poor job of communicating mathematics. Every year there are millions of people who take math courses. A large number of people who you meet on the street or on the airplane have taken college level mathematics. How many enjoyed it? How many saw no real point to it? While this book is not addressed to that random airplane person, it is addressed to beginning graduate students, people who already enjoy mathematics but who all too frequently get blown out of the mathematical water by mathematics presented in an unmotivated, but rigorous, manner. There is no problem with being nonrigorous, as long as you know and clearly label when you are being nonrigorous.

Comments on the Bibliography. There are many topics in this book. While I would love to be able to say that I thoroughly know the literature on each of these topics, that would be a lie. The bibliography has been cobbled together from recommendations from colleagues, from books that I have taught from and books that I have used. I am confident that there are excellent texts that I do not know about. If you have a favorite, please let me know at tgarrity@williams.edu.

While this book was being written, Paulo Ney De Souza and Jorge-Nuno Silva wrote *Berkeley Problems in Mathematics* [26], which is an excellent collection of problems that have appeared over the years on qualifying exams (usually taken in the first or second year of graduate school) in the math department at Berkeley. In many ways, their book is the complement of this one, as their work is the place to go to when you want to test your computational skills while this book concentrates on underlying intuitions. For example, say you want to learn about complex analysis. You should first read chapter nine of this book to get an overview of the basics about complex analysis. Then choose a good complex analysis book and work most of its exercises. Then use the problems in De Souza and Silva as a final test of your knowledge.

Finally, the book *Mathematics, Form and Function* by Mac Lane [82], is excellent. It provides an overview of much of mathematics. I am listing it here because there was no other place where it could be naturally referenced. Second and third year graduate students should seriously consider reading this book.

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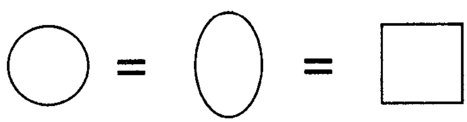
Finally, near the completion of this work, Bob Mizner passed away at an early age. It is in his memory that I dedicate this book (though no doubt he would have disagreed with most of my presentations and choices of topics; he definitely would have made fun of the lack of rigor).

On the Structure of Mathematics

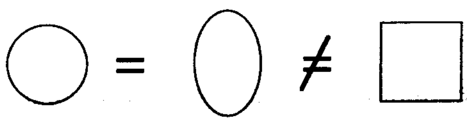
If you look at articles in current journals, the range of topics seems immense. How could anyone even begin to make sense out of all of these topics? And indeed there is a glimmer of truth in this. People cannot effortlessly switch from one research field to another. But not all is chaos. There are at least two ways of placing some type of structure on all of mathematics.

Equivalence Problems

Mathematicians want to know when things are the same, or, when they are equivalent. What is meant by the same is what distinguishes one branch of mathematics from another. For example, a topologist will consider two geometric objects (technically, two topological spaces) to be the same if one can be twisted and bent, but not ripped, into the other. Thus for a topologist, we have

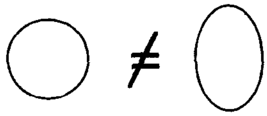


To a differential topologist, two geometric objects are the same if one can be smoothly bent and twisted into the other. By smooth we mean that no sharp edges can be introduced. Then



The four sharp corners of the square are what prevent it from being equivalent to the circle.

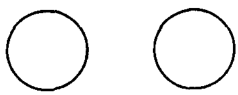
For a differential geometer, the notion of equivalence is even more restrictive. Here two objects are the same not only if one can be smoothly bent and twisted into the other but also if the curvatures agree. Thus for the differential geometer, the circle is no longer equivalent to the ellipse:



As a first pass to placing structure on mathematics, we can view an area of mathematics as consisting of certain *Objects*, coupled with the notion of *Equivalence* between these objects. We can explain equivalence by looking at the allowed *Maps*, or functions, between the objects. At the beginning of most chapters, we will list the Objects and the Maps between the objects that are key for that subject. The *Equivalence Problem* is of course the problem of determining when two objects are the same, using the allowable maps.

If the equivalence problem is easy to solve for some class of objects, then the corresponding branch of mathematics will no longer be active. If the equivalence problem is too hard to solve, with no known ways of attacking the problem, then the corresponding branch of mathematics will again not be active, though of course for opposite reasons. The hot areas of mathematics are precisely those for which there are rich partial but not complete answers to the equivalence problem. But what could we mean by a partial answer?

Here enters the notion of invariance. Start with an example. Certainly the circle, as a topological space, is different from two circles,



since a circle has only one connected component and two circles have two connected components. We map each topological space to a positive integer, namely the number of connected components of the topological space. Thus we have:

$$\text{Topological Spaces} \to \text{Positive Integers.}$$

The key is that the number of connected components for a space cannot change under the notion of topological equivalence (under bendings and twistings). We say that the number of connected components is an ***invariant*** of a topological space. Thus if the spaces map to different numbers, meaning that they have different numbers of connected components, then the two spaces cannot be topologically equivalent.

Of course, two spaces can have the same number of connected components and still be different. For example, both the circle and the sphere

have only one connected component, but they are different. (These can be distinguished by looking at each space's dimension, which is another topological invariant.) The goal of topology is to find enough invariants to be able to always determine when two spaces are different or the same. This has not come close to being done. Much of algebraic topology maps each space not to invariant numbers but to other types of algebraic objects, such as groups and rings. Similar techniques show up throughout mathematics. This provides for tremendous interplay between different branches of mathematics.

The Study of Functions

The mantra that we should all chant each night before bed is:

Functions describe the World.

To a large extent what makes mathematics so useful to the world is that seemingly disparate real-world situations can be described by the same type of function. For example, think of how many different problems can be recast as finding the maximum or minimum of a function.

Different areas of mathematics study different types of functions. Calculus studies differentiable functions from the real numbers to the real numbers, algebra studies polynomials of degree one and two (in high school) and permutations (in college), linear algebra studies linear functions, or matrix multiplication.

Thus in learning a new area of mathematics, you should always "find the function" of interest. Hence at the beginning of most chapters we will state the type of function that will be studied.

Equivalence Problems in Physics

Physics is an experimental science. Hence any question in physics must eventually be answered by performing an experiment. But experiments come down to making observations, which usually are described by certain computable numbers, such as velocity, mass or charge. Thus the experiments in physics are described by numbers that are read off in the lab. More succinctly, physics is ultimately:

Numbers in Boxes

where the boxes are various pieces of lab machinery used to make measurements. But different boxes (different lab set-ups) can yield different numbers, even if the underlying physics is the same. This happens even at the trivial level of choice of units.

More deeply, suppose you are modeling the physical state of a system as the solution of a differential equation. To write down the differential equation, a coordinate system must be chosen. The allowed changes of coordinates are determined by the physics. For example, Newtonian physics can be distinguished from Special Relativity in that each has different allowable changes of coordinates.

Thus while physics is 'Numbers in Boxes', the true questions come down to when different numbers represent the same physics. But this is an equivalence problem; mathematics comes to the fore. (This explains in part the heavy need for advanced mathematics in physics.) Physicists want to find physics invariants. Usually, though, physicists call their invariants 'Conservation Laws'. For example, in classical physics the conservation of energy can be recast as the statement that the function that represents energy is an invariant function.

Brief Summaries of Topics

0.1 Linear Algebra

Linear algebra studies linear transformations and vector spaces, or in another language, matrix multiplication and the vector space $\mathbf{R}^n$. You should know how to translate between the language of abstract vector spaces and the language of matrices. In particular, given a basis for a vector space, you should know how to represent any linear transformation as a matrix. Further, given two matrices, you should know how to determine if these matrices actually represent the same linear transformation, but under different choices of bases. The key theorem of linear algebra is a statement that gives many equivalent descriptions for when a matrix is invertible. These equivalences should be known cold. You should also know why eigenvectors and eigenvalues occur naturally in linear algebra.

0.2 Real Analysis

The basic definitions of a limit, continuity, differentiation and integration should be known and understood in terms of ϵ's and δ's. Using this ϵ and δ language, you should be comfortable with the idea of uniform convergence of functions.

0.3 Differentiating Vector-Valued Functions

The goal of the Inverse Function Theorem is to show that a differentiable function $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is locally invertible if and only if the determinant of its derivative (the Jacobian) is non-zero. You should be comfortable with what it means for a vector-valued function to be differentiable, why its derivative must be a linear map (and hence representable as a matrix, the Jacobian) and how to compute the Jacobian. Further, you should know the statement of the Implicit Function Theorem and see why is is closely related to the Inverse Function Theorem.

0.4 Point Set Topology

You should understand how to define a topology in terms of open sets and how to express the idea of continuous functions in terms of open sets. The standard topology on $\mathbf{R}^n$ must be well understood, at least to the level of the Heine-Borel Theorem. Finally, you should know what a metric space is and how a metric can be used to define open sets and hence a topology.

0.5 Classical Stokes' Theorems

You should know about the calculus of vector fields. In particular, you should know how to compute, and know the geometric interpretations behind, the curl and the divergence of a vector field, the gradient of a function and the path integral along a curve. Then you should know the classical extensions of the Fundamental Theorem of Calculus, namely the Divergence Theorem and Stokes' Theorem. You should especially understand why these are indeed generalizations of the Fundamental Theorem of Calculus.

0.6 Differential Forms and Stokes' Theorem

Manifolds are naturally occurring geometric objects. Differential k-forms are the tools for doing calculus on manifolds. You should know the various ways for defining a manifold, how to define and to think about differential k-forms, and how to take the exterior derivative of a k-form. You should also be able to translate from the language of k-forms and exterior derivatives to the language from Chapter Five on vector fields, gradients, curls and divergences. Finally, you should know the statement of Stokes' Theorem, understand why it is a sharp quantitative statement about the equality of the integral of a k-form on the boundary of a $(k+1)$-dimensional manifold with the integral of the exterior derivative of the k-form on the manifold, and how this Stokes' Theorem has as special cases the Divergence Theorem and the Stokes' Theorem from the previous chapter.

0.7 Curvature for Curves and Surfaces

Curvature, in all of its manifestations, attempts to measure the rate of change of the directions of tangent spaces of geometric objects. You should