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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

Preface

The purpose of this text is to offer a comprehensive and self-contained presentation of some of the most successful and popular domain decomposition methods for partial differential equations. Strong emphasis is put on both algorithmic and mathematical aspects. In addition, we have wished to present a number of methods that have not been treated previously in other monographs and surveys. We believe that this monograph will offer something new and that it will complement those of Smith, Bjørstad, and Gropp [424] and Quarteroni and Valli [392]. Our monograph is also more extensive and broader than the surveys given in Chan and Mathew [132], Farhat and Roux [201], Le Tallec [308], the habilitation thesis by Wohlmuth [469], and the well-known SIAM Review articles by Xu [472] and Xu and Zou [476].

Domain decomposition generally refers to the splitting of a partial differential equation, or an approximation thereof, into coupled problems on smaller subdomains forming a partition of the original domain. This decomposition may enter at the continuous level, where different physical models may be used in different regions, or at the discretization level, where it may be convenient to employ different approximation methods in different regions, or in the solution of the algebraic systems arising from the approximation of the partial differential equation. These three aspects are very often interconnected in practice.

This monograph is entirely devoted to the third aspect of domain decomposition. In practical applications, finite element or other discretizations reduces the problem to the solution of an often huge algebraic system of equations. Direct factorization of such systems might then not be a viable option and the use of basic iterative methods, such as the conjugate gradient algorithm, can result in very slow convergence. The basic idea of domain decomposition is that instead of solving one huge problem on a domain, it may be convenient (or necessary) to solve many smaller problems on single subdomains a certain number of times. Much of the work in domain decomposition relates to the selection of subproblems that ensure that the rate of convergence of the new iterative method is fast. In other words, domain decomposition methods provide preconditioners that can be accelerated by Krylov space methods.

The development of the field, and the increased interest in domain decomposition methods, is closely related to the growth of high speed computing. We note that in the June 2004 edition of the "Top 500" list, there are no fewer than 242 computer systems sustaining at least 1.0 Teraflop/sec. Scientific computing is therefore changing very fast and many scientists are now developing codes for parallel and distributed systems.

The development of numerical methods for large algebraic systems is central in the development of efficient codes for computational fluid dynamics, elasticity, and other core problems of continuum mechanics. Many other tasks in such codes parallelize relatively easily. The importance of the algebraic system solvers is therefore increasing with the appearance of new computing systems, with a substantial number of fast processors, each with relatively large memory. In addition, robust algebraic solvers for many practical problems and discretizations cannot be constructed by simple algebraic techniques, such as approximate inverses or incomplete factorizations, but the partial differential equation and the discretization must be taken into account. A very desirable feature of domain decomposition algorithms is that they respect the memory hierarchy of modern parallel and distributed computing systems, which is essential for approaching peak floating point performance. The development of improved methods, together with more powerful computer systems, is making it possible to carry out simulations in three dimensions, with quite high resolution, relatively easily. This work is now supported by high quality software systems, such as Argonne's PETSc library, which facilitates code development as well as the access to a variety of parallel and distributed computer systems. In chapters 6 and 9, we will describe numerical experiments with codes developed using this library.

A powerful approach to the analysis and development of domain decomposition is to view the procedure in terms of subspaces, of the original solution space, and with suitable solvers on these subspaces. Typically these subspaces are related to the geometrical objects of the subdomain partition (subdomains, subdomain boundaries, interfaces between subdomains, and vertices, edges, and faces of these interfaces). The abstract Schwarz theory, presented in Chap. 2, relies on these ideas and the convergence of the resulting iterative method is related to the stability of the decomposition into subspaces, certain stability properties of the local solvers, and a measure of the 'orthogonality' of these subspaces. The strong connection between stable decompositions of discrete functions in terms of Sobolev norms and the performance of the corresponding domain decomposition algorithm is not a mere way of giving an elegant mathematical description of a method that already works well in practice, but it is often the way in which new powerful algorithms are actually developed, especially for less standard discretizations such as edge elements for electromagnetic problems.

The book is addressed to mathematicians, computer scientists, and in general to people who are involved in the numerical approximation of partial differential equations, and who want to learn the basic ideas of domain decomposition methods both from a mathematical and an algorithmic point of view. The mathematical tools needed for the type of analysis essentially consist in some basic results on Sobolev spaces, which are reviewed in appendix A. The analysis also employs discrete Sobolev-type inequalities valid for finite element functions and polynomials. These tools are developed in Chap. 4. A basic knowledge of finite element theory and iterative methods is also required and two additional appendices summarize the results that are needed for the full understanding of the analysis of the algorithms developed in the main part of this monograph.

The literature of the field is now quite extensive and it has developed rapidly over the past twenty years. We have been forced to make some important omissions. The most important one is that we do not consider multilevel or multigrid methods, even though many of these algorithms can also be viewed, and then analyzed, using similar techniques as domain decomposition methods; the decomposition into subspaces is now related to a hierarchy of finite element meshes. The inclusion of these methods would have required a large effort and many pages and is likely to have duplicated efforts by real specialists in that field; the authors fully realizes the importance of these algorithms, which provide efficient and robust algorithms for many very large problems.

Other omissions have also been necessary:

- As already mentioned, we only consider domain decomposition as a way of building iterative methods for the solution of algebraic systems of equations.
- While we describe a number of algorithms in such a way as to simplify their implementation, we do not discuss other practical aspects of the development of codes for parallel and distributed computer systems.
- We only consider linear elliptic scalar and vector problems in full detail. Indeed, the methods presented in this monograph can be applied to the solution of linear systems arising from implicit time step discretizations of time-dependent problems or arising from Newton-type iterations for non-linear problems.
- Our presentation and analysis is mainly confined to low-order finite element (h version) and spectral element (a particular p version) approximations. Some domain decomposition preconditioners have also been applied to other types of p and to certain hp approximations and we only briefly comment on some of them in Sect. 7.5. We believe that many important issues remain to be addressed in this field.
- We have not touched the important problems of preconditioning plate and shell problems.

- Our presentation is restricted to conforming approximations. No preconditioner is presented for, e.g., mortar methods or other approximations on nonmatching grids.
- We have also been unable to cover the recent work on domain decomposition methods in time and space which has originated with work by Jacques-Louis Lions and Yvon Maday.

The authors wish to thank, besides the anonymous referees, the many friends that have gone over this monograph or part of it and provided us with important and helpful suggestions, references, and material. They are: Xiao-Chuan Cai, Maksymilian Dryja, Bernhard Hientzsch, Axel Klawonn, Rolf Krause, Frédéric Nataf, Luca Pavarino, Alfio Quarteroni, Marcus Sarkis, Christoph Schwab, Daniel Szyld, Xavier Vasseur, and last, but not least, Barbara Wohlmuth. We would also like to thank Charbel Farhat and Oliver Rheinbach for providing us with several figures.

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We end this preface by summarizing the contents of the various chapters in order to facilitate for the reader and to accommodate his/her specific interests.

In Chap. 1, *Introduction*, we present some basic ideas of domain decomposition. In particular, we show how matching conditions for traces and fluxes of the differential problems give rise to conditions on the finite element algebraic system, how simple subdomain iterations can be devised which contain many of the ideas employed by more recent and powerful preconditioners for large scale computations on many subdomains, and how some of the ideas employed in the discussion of the Schwarz alternating method and block Jacobi preconditioners naturally lead up to the abstract Schwarz theory. This is a chapter that requires little in terms of mathematical background. We recommend it to the reader who would like to understand the basic ideas of domain decomposition without entering the specifics of the more complicated, practical algorithms. The last section, Sect. 1.6 contains some less standard and earlier results on overlapping methods and can be bypassed initially.

Chapter 2, *Abstract Theory of Schwarz Methods*, contains the standard abstract theory of additive and multiplicative Schwarz algorithms, together with some additional topics, such as coloring arguments and some hybrid algorithms. The three basic ideas of stable decompositions, strengthened Cauchy inequalities, and stable local solvers contained in three assumptions in Sect.

2.3 are central and therefore recommended in order to prepare for the chapters that follow.

Chapter 3, *Overlapping Methods*, presents overlapping preconditioners in a more general way than is normally done, since we allow for general coarse meshes and problems. In addition, the chapter contains a section on scaling and quotient space arguments, which are routinely employed in the analysis of domain decomposition preconditioners. The sections on restricted algorithms and alternative coarse problems can be bypassed initially.

In Chap. 4, *Substructuring Methods: Introduction*, we present the basic ideas of iterative substructuring methods, which are based on nonoverlapping partitions into subdomains, interior and interface variables, vertex, edge and face variables, Schur complement systems, and discrete harmonic extensions. These notions, at least at a basic level, are necessary in order to understand the iterative substructuring methods developed in the chapters that follow. The last section, Sect. 4.6 contains the Sobolev type inequalities necessary to fully analyze iterative substructuring methods and is necessary for the reader who also wishes to understand the proofs in the chapters that follow.

Chapter 5 is devoted to *Primal Iterative Substructuring Methods* for problems in three dimensions. In Sect. 5.3, we first treat the problem of devising effective local solvers by decoupling degrees of freedom associated with the vertices, edges, and faces of the subdomain partition. In Sect. 5.4, we then consider the problems of devising efficient coarse solvers, which are the key and a quite delicate part of any successful preconditioners for three-dimensional problems.

Chapter 6 is devoted to *Neumann-Neumann and FETI Methods*. We have decided to present these algorithms and their analysis together; recent research has established more and more connections between the two classes of methods. A key ingredient of this analysis is the stability of certain average and interface jump operators (cf. Sect. 6.2.3 and 6.4.3). One of the purposes of this chapter is to present the basics of one-level FETI and the more recent FETI-DP algorithms in a self-contained, sufficiently deep manner. For the reader who is interested in only the basic ideas of these methods, we recommend Sect. 6.3.1 for one-level FETI and Sect. 6.4.1, where the important ideas of FETI-DP can already be appreciated and understood in the more intuitive two-dimensional case.

In Chap. 7, we present generalizations to *Spectral Element Methods*. A basic knowledge of the corresponding algorithms for the h version in the previous chapters is required. The fundamental equivalence between spectral element approximations and some finite element approximations on Gauss-Lobatto meshes is the key ingredient for the development and analysis of both overlapping and nonoverlapping methods; this is the idea underlying the Deville-Mund preconditioners reviewed in Sect. 7.2. Only those parts that are different from the proofs of the h version are treated explicitly in this chapter. We have also added a brief discussion and review of domain decomposition for more general p and hp version finite elements with references to the literature.

In Chap. 8, generalizations to *Linear Elasticity* problems are considered. A basic knowledge of the corresponding algorithms for scalar problems in the previous chapters is required and only those parts that are different from the scalar case are treated explicitly.

In Chap. 9, some selected topics on *Preconditioners for Saddle Point Problems* are presented. They are: some basic ideas about preconditioning saddle-point problems, block-diagonal and block-triangular preconditioners, and some overlapping and iterative substructuring methods. We primarily consider the Stokes system (and briefly the related problem of incompressible elasticity) and flow in porous media. As a general rule, we only review the basic results and refer the reader to the appropriate references for more detailed and thorough presentations.

Chapter 10 is devoted to the field of domain decomposition preconditioners for *Problems in* $H(\mathrm{div}\,;\Omega)$ *and* $H(\mathbf{curl}\,;\Omega)$, which has developed relatively recently. This chapter requires a basic knowledge of the corresponding algorithms for scalar problems. Here, proofs are presented in full detail and this chapter is intended as a self-contained and deep treatment of domain decomposition methods for these problems, the analysis and development of which is in general more technically demanding than for more standard scalar and vector problems. Sections 10.1.1, 10.1.2, and 10.2.1, in particular, contain the technical tools necessary for the analysis and can be bypassed by a reader who is only interested in understanding the algorithms.

Chapter 11 is devoted to *Indefinite and Nonsymmetric Problems.* We first present a generalization of the abstract Schwarz theory to nonsymmetric and/or indefinite problems in detail. We also present some selected topics on domain decomposition preconditioners which are commonly employed in large scale computations but for which very little theory is available. These are algorithms for convection-dominated scalar problems, the Helmholtz equations, eigenvalue and nonlinear problems. This part is only intended as an overview and to provide a collection of relevant references to the literature.

The volume ends with three appendices, references, and an index.

Zürich, New York,
July 2004

Andrea Toselli
Olof Widlund

Contents

1

Introduction

1.1 Basic Ideas of Domain Decomposition

The basic ideas of domain decomposition are quite natural and simple. Consider the Poisson equation on a region Ω, in two or three dimensions, with zero Dirichlet data given on $\partial\Omega$, the boundary of Ω. Suppose also that Ω is partitioned into two nonoverlapping subdomains Ω_i :

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma = \partial\Omega_1 \cap \partial\Omega_2;$$

see Fig. 1.1. We also assume that

$$\text{measure}(\partial\Omega_1 \cap \partial\Omega) > 0, \quad \text{measure}(\partial\Omega_2 \cap \partial\Omega) > 0,$$

and that the boundaries of the subdomains are Lipschitz continuous, and consider the following problem:

$$\begin{aligned} -\Delta u &= f \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Under suitable regularity assumptions on f and the boundaries of the subdomains, typically f square-summable and the boundaries Lipschitz, problem (1.1) is equivalent to the following coupled problem:

$$\begin{aligned} -\Delta u_1 &= f \qquad && \text{in } \Omega_1, \\ u_1 &= 0 && \text{on } \partial\Omega_1 \setminus \Gamma, \\ u_1 &= u_2 && \text{on } \Gamma, \\ \frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} && \text{on } \Gamma, \\ -\Delta u_2 &= f && \text{in } \Omega_2, \\ u_2 &= 0 && \text{on } \partial\Omega_2 \setminus \Gamma. \end{aligned} \tag{1.2}$$

Here u_i is the restriction of u to Ω_i and $\mathbf{n}_i$ the outward normal to Ω_i. This equivalence can be proven by considering the corresponding variational problems; see [392, Sect. 1.2]. The conditions on the interface Γ are called *transmission conditions* and they are also equivalent to the equality of any two independent linear combinations of the traces of the functions and their normal derivatives. In the following, we will also refer to the normal derivative as the *flux*.

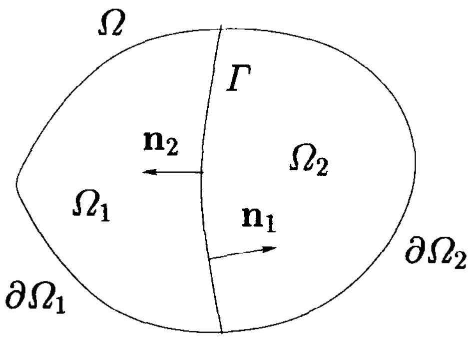

Fig. 1.1. Partition into two nonoverlapping subdomains.

Remark 1.1. The following one-dimensional example shows that some regularity beyond $f \in H^{-1}(\Omega)$ is required. Let u be the weak solution of

$$\begin{aligned} -\frac{d^2u}{dx^2} &= -2\,\delta \qquad \text{in } (-1,1), \\ u(-1) = u(1) &= 0, \end{aligned} \tag{1.3}$$

where $\delta(x)$ is the delta function. The unique weak solution $u \in H_0^1(-1,1)$ is

$$u(x) = \begin{cases} -1 - x & x \leq 0, \\ -1 + x & x \geq 0, \end{cases}$$

and its derivative has a jump at $x = 0$.

We note that this particular problem is quite relevant to domain decomposition theory. In many algorithms, we will first eliminate all nonzero components of the right hand side, of a finite element approximation, except those on the interface, in this case $x = 0$. We are then left with an equation for the remaining finite element error, which is a direct analog of equation (1.3).

1.2 Matrix and Vector Representations

In this section, we consider matrix and vector representations of certain operators and linear functionals; we refer to appendix B for additional details.

Starting with any domain decomposition algorithm written in terms of functions and operators, we will be able to rewrite it in matrix form as a preconditioned iterative method for a certain linear system.

We now consider a triangulation of the domain Ω and a finite element approximation of problem (1.1). We always assume that subdomains consist of unions of elements or, equivalently, that subdomain boundaries do not cut through any elements. Such an approximation gives rise to a linear system

$$Au = f \tag{1.4}$$

with a symmetric, positive definite matrix which, for a mesh size of h, typically has a condition number on the order of $1/h^2$. Here,

$$A = \begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{pmatrix}, \quad u = \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix}, \quad f = \begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{pmatrix}, \tag{1.5}$$

where we have partitioned the degrees of freedom into those internal to Ω_1, and to Ω_2, and those of the interior of Γ.

The stiffness matrix A and the load vector f can be obtained by subassembling the corresponding components contributed by the two subdomains. Indeed, if

$$f^{(i)} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}, \quad A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix}, \quad i = 1, 2, \tag{1.6}$$

are the right hand sides and the local stiffness matrices for Poisson problems with a Dirichlet condition on $\partial\Omega_i \setminus \Gamma$ and a Neumann condition on Γ, we have

$$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}, \quad f_\Gamma = f_\Gamma^{(1)} + f_\Gamma^{(2)}.$$

In view of the transmission conditions in (1.2), we will look for an *approximation* of the normal derivatives on Γ. Given the local exact solution u_i, its normal derivative can be defined as a linear functional by using Green's formula. Thus, if ϕ_j is a nodal basis function for a node on Γ, we have, using (1.2),

$$\int_\Gamma \frac{\partial u_i}{\partial n_i}\phi_j \, ds = \int_{\Omega_i} (\Delta u_i \phi_j + \nabla u_i \cdot \nabla \phi_j) \, dx = \int_{\Omega_i} (-f\phi_j + \nabla u_i \cdot \nabla \phi_j) \, dx.$$

An approximation, $\lambda^{(i)}$, of the functional representing the normal derivative can be found by replacing the exact solution u_i in the right hand side with its finite element approximation. Letting j run over the nodes on Γ and using the definition of the local stiffness matrix, we introduce the expression

$$\lambda^{(i)} = A_{\Gamma I}^{(i)} u_I^{(i)} + A_{\Gamma\Gamma}^{(i)} u_\Gamma^{(i)} - f_\Gamma^{(i)}. \tag{1.7}$$