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COEFFICIENT REGIONS FOR SCHLICHT FUNCTIONS

BY
A. C. SCHAEFFER
AND
D. C. SPENCER

WITH A CHAPTER ON

THE REGION OF VALUES OF THE DERIVATIVE OF A SCHLICHT FUNCTION

BY
ARTHUR GRAD

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PREFACE

Instead of investigating various isolated extremal problems in the theory of schlicht functions, the authors have concentrated their efforts during the last three years on the investigation of the family of extremal schlicht functions in the large and this monograph is a presentation of the results of this research. For the sake of completeness and readability it has been found desirable to include in some places work that has been published elsewhere by ourselves or others. As most of the material is new, we have tried to point out carefully the material which already exists in published form.

In the calculus of variations there are two classical approaches: (a) study of specific problems using local variations; (b) study of a whole class of extremal problems and the investigation of the structure of the class as a whole. Variational methods in conformal mapping have been developed systematically in the last few years, beginning with a paper by M. Schiffer in 1938. The various publications on this subject during the last ten years have been mainly concerned with results of type (a) whereas we have tried to develop in this monograph a systematic approach to results of type (b), but we do not believe that our approach is the only one.

Since the investigation of extremal problems in conformal mapping embraces a rather wide field of research, we have confined ourselves to extremal problems relating to a finite number of the coefficients in the Taylor expansion of a function which is regular and schlicht inside the unit circle. Results of type (b) then concern the study of the region of values of the first n coefficients considered as a point in multi-dimensional euclidean space. This problem is only one of a host of problems that can be formulated in the theory of schlicht functions, and indeed a much more general problem is mentioned in Chapter I. The authors have chosen to investigate the coefficient problem not only because of its classical interest but also because it seems likely that the methods developed in this special case can be extended to many other problems. Dr. A. Grad has added a chapter in which he investigates the region of possible values of the derivative of a schlicht function at a fixed point inside the unit circle and his solution provides another example of these methods. A somewhat different version of his work has already appeared in hektographed form.

We have tried to make this monograph self-contained to as large an extent as practicable, and for this reason we have tried to keep the proofs and phraseology on as elementary a level as possible. This has lengthened the proofs in only a few cases and altogether it has increased the total length only slightly. We feel that sufficient background for reading this monograph is provided by a knowledge which is comparable to that contained in standard books on the theory of functions.

October, 1948.

A. C. SCHAEFFER and D. C. SPENCER
Purdue University and Stanford University

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September, 1950

A. C. Schaeffer and D. C. Spencer
University of Wisconsin and Princeton University

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Chapter I

HISTORY OF SCHLICHT FUNCTIONS AND ELEMENTARY PROPERTIES OF THE nTH REGION

1.1. We begin with a brief history of the theory of schlicht functions, and we mention only those results which bear directly on the coefficient problem or on the problem of the region of values of $f'(z)$.

A function $f(z)$ is said to be schlicht in a domain if for any two points z_1 and z_2 of it we have $f(z_1) = f(z_2)$ only if $z_1 = z_2$. We shall be concerned with functions which are regular and schlicht in the unit circle $|z| < 1$ and which are normalized by the condition that the function vanishes at the origin and has a first derivative there equal to 1. The class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are regular and schlicht in $|z| < 1$ will be denoted by $\mathfrak{S}$.

The starting point in the investigation of schlicht functions was a paper by P. Koebe in 1907 on the uniformization of algebraic curves (see [10])[1] in which he proved, in particular, that there is a constant k (Koebe's constant) such that the boundary of the map of $|z| < 1$ by any function $w = f(z)$ of class $\mathfrak{S}$ is always at a distance not less than k from $w = 0$. A related result is that there exist bounds for the modulus of the derivative of $f(z)$ at any point in $|z| < 1$, these bounds depending only on $|z|$. These properties may be derived from the fact that the family $\mathfrak{S}$ is compact or, in other words, that $\mathfrak{S}$ is a normal family in the sense of Montel [15]. Actually, with the introduction of some such metric as

$$d(f_1, f_2) = \sup_{|z| = 1/2} |f_1 - f_2|$$

$\mathfrak{S}$ becomes a compact metric space.

Koebe's result soon attracted the attention of others (Plemelj [18]; Gronwall [7b, c]; Pick [17]; Faber [5]; Bieberbach [1]). Gronwall [7a] first gave the so-called "area-principle" which asserts that if the function

$$g(z) = 1/z + \sum_{\nu=1}^{\infty} b_\nu z^\nu$$

is schlicht in $|z| < 1$ and regular except at $z = 0$ where there is a simple pole, then

$$\sum_{\nu=1}^{\infty} \nu |b_\nu|^2 \leqq 1.$$

The image of $|z| < r$, $r < 1$, by $w = g(z)$ leaves uncovered a certain domain

[1] Square brackets refer to the bibliography at the end of the book.

of the w-plane, and the area-principle is an expression of the fact that the area of this domain is positive. Gronwall's paper seems to have attracted little or no attention, but in 1916 the area-principle was rediscovered and used to obtain the precise values of the constants in Koebe's results ([1], [5]). It was found that $k = 1/4$ and that

$$\frac{1 - |z|}{(1 + |z|)^3} \leqq |f'(z)| \leqq \frac{1 + |z|}{(1 - |z|)^3}. \tag{1.1.1}$$

Here the validity of the upper bound for all $|z| < 1$ gives the precise inequality[2]

$$|a_2| \leqq 2, \tag{1.1.2}$$

a result which was proved at the same time. Equality occurs in (1.1.1) (either side) or in (1.1.2) only for the function (Koebe function)

$$f(z) = \frac{z}{(1 + e^{i\varphi}z)^2}, \qquad \varphi \text{ real.} \tag{1.1.3}$$

Because of the extremal character of the function (1.1.3) so far as the inequalities (1.1.1), (1.1.2), and some others are concerned and because the nth coefficient of this function is equal to n in modulus, it was conjectured in 1916 or shortly thereafter that

$$|a_n| \leqq n, \qquad n = 2, 3, \cdots.$$

It was approximately at this time that Bieberbach proposed the so-called coefficient problem for schlicht functions. This is the problem of finding for each n, $n \geqq 2$, the precise region V_n in euclidean space of $2n - 2$ real dimensions occupied by points $(a_2, a_3, \cdots, a_n)$ corresponding to functions of class $\mathfrak{S}$. Several years earlier Carathéodory [2] and others had focused interest on this type of question by solving the coefficient problem for functions

$$p(z) = 1 + \sum_{\nu=1}^{\infty} c_\nu z^\nu \tag{1.1.4}$$

which are regular in $|z| < 1$ and have positive real parts there.

The paper [13] of Löwner forms a landmark in the historic development of this subject. Löwner gave a representation of the coefficients a_ν of a class of schlicht functions which lie everywhere dense in $\mathfrak{S}$, the representation being in terms of integrals of a function $\kappa(t)$, $|\kappa(t)| = 1$. For example, the formulas for a_2, a_3 are:

$$a_2 = -2 \int_0^\infty e^{-\tau} \kappa(\tau)\, d\tau, \tag{1.1.5}$$

$$a_3 = -2 \int_0^\infty e^{-2\tau} \kappa(\tau)^2\, d\tau + 4 \left(\int_0^\infty e^{-\tau} \kappa(\tau)\, d\tau \right)^2. \tag{1.1.6}$$

[2] On the other hand, (1.1.2) implies (1.1.1) (both the upper and lower bounds). See [12b].

It follows at once from (1.1.5) that $|a_2| \leqq 2$, and it is readily shown from (1.1.6) (as Löwner pointed out) that

$$|a_3| \leqq 3. \tag{1.1.7}$$

The inequality (1.1.7) seems to be beyond the power of the methods used by Löwner's predecessors.

In 1925 Littlewood [12a] proved that

$$|a_n| < e \cdot n, \qquad n = 2, 3, \cdots, \tag{1.1.8}$$

and two years later Prawitz [19] gave a generalization of the area-principle. The method used by Littlewood may also be regarded as a somewhat different extension of the area-principle.

So far as the coefficient problem is concerned, mention must be made of the paper by Rogosinski [21] in which he introduced the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are regular in $|z| < 1$ and have the property that they assume real values if and only if z is real. Rogosinski called such functions typically-real. All coefficients of a typically-real function are real and Im (f) and Im (z) have the same sign in $|z| < 1$. Schlicht functions of $\mathfrak{S}$ with all coefficients real form probably the most important subclass of these functions. We say that a class of functions is convex if, given any two functions f_1, f_2 of the class, the weighted mean

$$\frac{\lambda_1 f_1 + \lambda_2 f_2}{\lambda_1 + \lambda_2}$$

also belongs to the class no matter how the positive weights λ_1, λ_2 are chosen. Typically-real functions clearly form a convex class, and so do the functions (1.1.4) having positive real parts. The relation between these two classes is extremely simple. In fact, if f is typically-real, then

$$p(z) = \frac{1 - z^2}{z} f(z) \tag{1.1.9}$$

is a function of positive real part with real coefficients and conversely. Hence the coefficients c_k of p are connected with the coefficients a_k of f by the formulas

$$c_k = a_{k+1} - a_{k-1}, \qquad a_k = \begin{cases} c_1 + c_3 + \cdots + c_{k-1}, & k \text{ even}, \\ 1 + c_2 + c_4 + \cdots + c_{k-1}, & k \text{ odd}. \end{cases} \tag{1.1.10}$$

Thus the nth coefficient region of typically-real functions (that is to say, the region of points $(a_2, a_3, \cdots, a_n)$ belonging to these functions) is a simple linear map of the region of points $(c_1, c_2, \cdots, c_{n-1})$. Since the latter region is known (see [2]), the coefficient problem for typically-real functions is solved. We remark that the nth coefficient region for typically-real functions is the smallest convex region containing the nth coefficient region of schlicht functions with real coefficients. For typically-real functions we have (as Rogosinski [21] pointed out)

$$|a_n| \leq n, \qquad n = 2, 3, \cdots, \tag{1.1.11}$$

and this estimate is a fortiori true of schlicht functions with real coefficients.[3] Equality is attained for the function (1.1.3) where $\varphi = 0$ or π.

Star-like schlicht functions also form an important subclass of $\mathfrak{S}$. A star-like schlicht function $w = f(z)$ maps $|z| < 1$ onto a domain in the w-plane having the property that any point of it can be connected to the origin $w = 0$ by a straight line lying entirely in the interior. A necessary and sufficient condition for $f(z)$ of class $\mathfrak{S}$ to be star-like is that

$$p(z) = z\frac{f'(z)}{f(z)}$$

be a function with real part positive in $|z| < 1$. Thus the coefficient regions V_n^* of star-like schlicht functions are also connected in a simple way with the coefficient regions of functions (1.1.4) having positive real part, and so the coefficient problem for these functions is solved. For star-like functions the estimate (1.1.11) is true, equality being attained for the star-like function (1.1.3) (see [6b]).

The coefficient problem for schlicht functions was first seriously considered by Peschl [16] in 1937. Peschl considered curves of Löwner type issuing from boundary points of the nth region V_n and extending to points of V_n^*, the coefficient region of star-like functions. He obtained qualitative results concerning V_n and also found the region of (a_2, a_3) when both a_2, a_3 are real. In the following years variational methods were introduced into the theory and new tools were thus provided ([6c,d,e], [22], [23], [24], and [25b]). Elementary variations were applied by Marty [14] to obtain one or two necessary conditions for functions $w = f(z)$ of class $\mathfrak{S}$ maximizing $|a_n|$. The systematic investigation of schlicht functions by the variational method began, however, with the paper by Schiffer [24a].

For many years Löwner's method provided one of the most powerful attacks in investigating schlicht functions. In recent years the variational approach has been developed to a point where it is comparable to Löwner's method in effectiveness and in some cases it seems to have led further. For example, not only has the variational method yielded many of the important results previously obtained only from Löwner's method (see [22a,b]), but it has also led to new results such as the regions of variability discussed below (see also [22c,f]). However, the variational approach has by no means displaced Löwner's method; rather it has complemented it. Löwner's integral representation for the coefficients,

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[3] The estimate (1.1.11) for functions of $\mathfrak{S}$ with real coefficients was given independently and almost simultaneously by Dieudonné [4], Rogosinski [21], and Szász [26].

as well as the condition of Dieudonné [4] and the interesting set of conditions given by Grunsky [8] (see also [24f]), may be interpreted as giving necessary and sufficient conditions on a set of numbers $(a_2, a_3, \cdots, a_n)$ in order that they should be the coefficients of a function of class $\mathfrak{S}$, the conditions being expressed in terms of infinitely many parameters. The method developed here expresses conditions in terms of finitely many parameters, as we shall show.

The variational method gives necessary conditions in order that a function $w = f(z)$ of class $\mathfrak{S}$ should extremalize an arbitrary function of its first n coefficients. The method of Teichmüller [27] complements this result by proving that the necessary conditions are sufficient.

The variational method is not only applicable to problems involving the coefficients but is also applicable to a wide class of problems in conformal mapping. These more general problems are briefly described in **1.3** below and are discussed in greater detail in [22d]. The authors have confined themselves mainly to the coefficient problem although similar methods are applicable, at least in principle, to the wider class of problems discussed in **1.3.**

One of the oldest problems concerning functions of class $\mathfrak{S}$ is that of determining the possible values of $f'(z_1)$ at a fixed point z_1, $|z_1| < 1$. As mentioned above (formula (1.1.1)), precise bounds for $|f'(z_1)|$ were found in 1916. In 1936 Golusin [6a], using Löwner's method, found the precise bounds for $|\arg f'(z)|$ when f belongs to $\mathfrak{S}$, namely

$$(1.1.1)' \qquad |\arg f'(z)| \leqq \begin{cases} 4 \arcsin |z|, & |z| \leqq \dfrac{1}{2^{1/2}}, \\ \pi + \log \dfrac{|z|^2}{1 - |z|^2}, & \dfrac{1}{2^{1/2}} < |z| < 1. \end{cases}$$

Here the multi-valued functions may be assumed to have their principal values. The inequality (1.1.1)′ complements (1.1.1) and constitutes the so-called "rot-a tion theorem" for schlicht functions. The method employed by Golusin to prove (1.1.1)′ was later systematized by Robinson [20], who used it in finding the region of points $(|f(z)|, |f'(z)|)$, z fixed.

It is convenient to consider the region of values of $f'(z_1)$ in the plane of log $f'(z_1)$. The inequalities (1.1.1) and (1.1.1)′ (translated to the logarithmic plane) place this region in a rectangle. In [22d] the authors derived a differential equation for functions $f(z)$ whose derivative at the point z_1 lies on the boundary of the region of values. This differential equation involves essentially one real parameter, and its solutions may be implicitly expressed in terms of elementary functions. Dr. Grad has added a chapter (Chapter XV) in which he determines the region of values of $f'(z_1)$, z_1 any fixed interior point of the unit circle. Dr. Grad's solution determines the exact region inside the above rectangle which is occupied by values $\log f'(z_1)$, z_1 fixed.

1.2. Before proceeding with the discussion of the coefficient regions V_n, a more precise definition should be given. The point $(a_2, a_3, \cdots, a_n)$ is said to belong to the region V_n in $(2n - 2)$-dimensional real euclidean space with coordinates Re (a_2), Im (a_2), $\cdots$, Re (a_n), Im (a_n) if there is a function

$$f(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu$$

of class $\mathfrak{S}$ such that

$$b_\nu = a_\nu, \qquad \nu = 2, 3, \cdots, n.$$

We say that the point belongs to the function and that the function belongs to the point. Only the region V_2 is given from earlier results; it is simply the circle

$$|a_2| \leqq 2.$$

In the following pages it will be shown that for any n, $n \geqq 2$, the boundary of V_n can be expressed in terms of finitely many parameters. In fact, the boundary will be dissected into finitely many portions $\Pi_1, \Pi_2, \cdots, \Pi_N$ and the coordinates a_k of the point $(a_2, a_3, \cdots, a_n)$ on any one of these portions Π_k will be functions of a finite number of parameters. The number of parameters defining Π_k will not exceed $2n - 3$. If the number of parameters is equal to $2n - 3$, then Π_k is a hypersurface of dimension $2n - 3$. In some cases, however, Π_k will depend on fewer than $2n - 3$ parameters; and if this is the case, Π_k will be a manifold of lower dimension which represents, for example, the intersection of two or more manifolds of higher dimension. In addition, we shall investigate certain geometric properties of the regions V_n.

In the case of the region V_3, there are essentially two hypersurfaces Π_1, Π_2 of dimension 3 which together with their 2-dimensional intersection make up the boundary. The parametric formulas for Π_1 and Π_2 are in terms of elementary functions. Tables have been computed for the boundary of the region V_3 and are included in the Appendix. This region is 4-dimensional, but it has a rotational property that makes it possible to define its entire structure from certain 3-dimensional cross-sections. If

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is a function of class $\mathfrak{S}$, then the functions

$$(1.2.1) \qquad e^{-i\theta} f(e^{i\theta} z) = z + a_2 e^{i\theta} z^2 + a_3 e^{2i\theta} z^3 + \cdots, \qquad \theta \text{ real},$$

and

$$(1.2.2) \qquad \bar{f}(\bar{z}) = z + \bar{a}_2 z^2 + \bar{a}_3 z^3 + \cdots,$$

$$\bar{a}_k = \text{complex conjugate of } a_k,$$

also belong to class $\mathfrak{S}$. Thus if any one of the points

$$(a_2, a_3), (a_2 e^{i\theta}, a_3 e^{2i\theta}), (\bar{a}_2, \bar{a}_3)$$

belongs to V_3, so do the others; and, in particular, the entire domain can be constructed from any cross-section arg (a_2) = constant or arg (a_3) = constant. If $(\text{Re}(a_2), \text{Re}(a_3), \text{Im}(a_3))$ is a point of the cross-section $\text{Im}(a_2) = 0$, then by a rotation (1.2.1) with $\theta = \pi$ and by a reflection (1.2.2) it is seen that the points $(-\text{Re}(a_2), \text{Re}(a_3), \text{Im}(a_3))$ and $(\text{Re}(a_2), \text{Re}(a_3), -\text{Im}(a_3))$ also belong to this cross-section. Thus the cross-section $\text{Im}(a_2) = 0$ is symmetric about the planes $\text{Re}(a_2) = 0$ and $\text{Im}(a_3) = 0$.

Table I gives the part of the boundary of the cross-section $\text{Im}(a_2) = 0$ which lies in

$$\text{Re}(a_2) \geqq 0, \text{Im}(a_3) \geqq 0.$$

Plate I on page xii shows the corresponding solid. It is one-half of the entire cross-section $\text{Im}(a_2) = 0$. The yellow and blue surfaces correspond to Π_1 and Π_2. The function $w = f(z)$ belonging to a point of the yellow surface Π_1 maps $|z| < 1$ onto the w-plane minus a single curved analytic slit, whereas the function $w = f(z)$ belonging to a point of the blue surface Π_2 maps $|z| < 1$ onto the w-plane minus a ray arg (w) = constant extending from $w = \infty$ to some finite point where there is a fork composed in general of two prongs which form angles $2\pi/3$ with the ray. We remark that to any boundary point of the region V_3 (more generally of the region V_n) there corresponds a unique boundary function $w = f(z)$; that is, boundary points and boundary functions correspond in a one–one way.

Table II gives the part of the boundary of the cross-section $\text{Im}(a_3) = 0$ which lies in

$$\text{Re}(a_2) \geqq 0, \text{Im}(a_2) \geqq 0,$$

and this is one-fourth of the boundary since the cross-section $\text{Im}(a_3) = 0$ is symmetric about the planes $\text{Re}(a_2) = 0$ and $\text{Im}\ (a_2) = 0$. Plate II on page xiv shows one-half of the entire cross-section $\text{Im}(a_3) = 0$, namely the half $\text{Re}(a_2) \geqq 0$.

A detailed discussion of the domain V_3 is given in Chapter XIII.

We remark that if we know the regions V_n, then the coefficient problem is solved for any simply-connected domain D containing $z = 0$. The coefficient problem for a domain D containing $z = 0$ is that of finding the regions of values of the coefficients in the development about $z = 0$ of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.2.3}$$

which are regular and schlicht in D. If D is the full plane or the plane punctured at one point, there is at most one function $f(z)$ which is regular and schlicht in D and of the form (1.2.3) near $z = 0$. For example, if D is the plane minus the point $z = 1$, the only schlicht function of form (1.2.3) near $z = 0$ is $f(z) = z/(1 - z)$. If D is not the full plane or the punctured plane, there is a function $z = \varphi(\zeta)$ which maps $|\zeta| < 1$ onto D with $\zeta = 0$ going into $z = 0$. Near $\zeta = 0$ let

$$z = \varphi(\zeta) = \beta_1\zeta + \beta_2\zeta^2 + \beta_3\zeta^3 + \cdots, \qquad \beta_1 > 0. \tag{1.2.4}$$

Writing

$$\{\varphi(\zeta)\}^{\kappa} = \sum_{\nu=\kappa}^{\infty} \beta_\nu^{(\kappa)} \zeta^\nu,$$

we have

$$g(\zeta) = f(\varphi(\zeta)) = \sum_{\nu=1}^{\infty} b_\nu \zeta^\nu$$

where

(1.2.5) $$b_\nu = \beta_\nu^{(1)} + \beta_\nu^{(2)} a_2 + \cdots + \beta_\nu^{(\nu)} a_\nu, \qquad \nu = 1, 2, \cdots$$

Since the coefficients β_ν (and so the $\beta_\nu^{(\kappa)}$) are given and since we know all possible values for the numbers $b_\nu/b_1 = b_\nu/\beta_1$, by hypothesis, the coefficient regions of the a_ν can be determined from (1.2.5).

1.3. The coefficient problem—that is, the problem of finding the domains V_n—is only one of a wide class of problems concerning the family $\mathfrak{S}$ of schlicht functions. In fact, let R be a closed set lying in $|z| < 1$ and let $\psi_\nu(\tau)$ be a measure function defined in the space R (see [22d]). Given an integer n, there is a number $M = M(n)$ such that

(1.3.1) $$|f^{(\nu)}(z)| \leq M, \qquad \nu = 0, 1, 2, \cdots, n,$$

for all z of R. Here $f^{(\nu)}(z)$ denotes the νth derivative of a function f of class $\mathfrak{S}$. The inequality (1.3.1) is a consequence of the fact that the class $\mathfrak{S}$ is compact. Let $F_\nu(\zeta_0, \bar{\zeta}_0, \cdots, \zeta_n, \bar{\zeta}_n)$ denote a complex-valued function which is continuous together with its first order partial derivatives in an open set containing the closed set $|\zeta_\nu| \leq M$ ($\nu = 0, 1, 2, \cdots, n$). Given the functions $F_1, F_2, \cdots, F_m$ and the measure functions $\psi_1, \psi_2, \cdots, \psi_m$, let

(1.3.2) $$P_\nu = \int_R F_\nu(f(z), \overline{f(z)}, \cdots, f^{(n)}(z), \overline{f^{(n)}(z)})\, d\psi_\nu,$$

for $\nu = 1, 2, \cdots, m$. If $f(z)$ belongs to $\mathfrak{S}$, the point

$$P_f = (P_1, P_2, \cdots, P_m)$$

is a point in a euclidean space of $2m$ real dimensions, and the point P_f is said to belong to $f(z)$. As f ranges over $\mathfrak{S}$, the point P_f belonging to f ranges over a set which we call D_n. Since $\mathfrak{S}$ is compact, the set D_n is closed and bounded. The problem is to find the region D_n.

Many of the problems on schlicht functions which concern the values taken in the interior of the unit circle, as opposed to boundary-value problems, are contained in this general formulation. The nth coefficient region V_n is a particular region D_n, and the region of values of the derivative $f'(z_1)$ at a fixed point z_1 in $|z| < 1$ is a particular region D_1. Although we shall discuss here only the special problem of the regions V_n and Dr. Grad in Chapter XV will discuss the region of values of $f'(z_1)$, we remark that the methods used are applicable at least to some extent to any problem contained in the above general formulation. In the cases where P_ν as defined by (1.3.2) reduces to the form

$$P_\nu = \sum_{j=1}^{k} F_\nu(f(z_j), \overline{f(z_j)}, \cdots, f^{(n)}(z_j), \overline{f^{(n)}(z_j)}),$$

the methods seem to be closely related to those used in the coefficient problem.

1.4. Before taking up the investigation of the domains V_n, we note certain of their elementary properties.

LEMMA I. *The following statements are equivalent:*

(i) $(a_2, a_3, \cdots, a_n)$ *is an interior point of* V_n;

(ii) *there is a bounded function of class* $\mathfrak{S}$ *belonging to the point* $(a_2, a_3, \cdots, a_n)$;

(iii) *there is a function* $w = f(z)$ *of class* $\mathfrak{S}$ *belonging to the point* $(a_2, a_3, \cdots, a_n)$, *the closure of whose values* w *in* $|z| < 1$ *does not fill the whole closed* w*-plane.*

PROOF: If (i) is true, there is an $\epsilon > 0$ such that all points $(c_2, c_3, \cdots, c_n)$ satisfying the inequality

$$\sum_{\nu=2}^{n} |c_\nu - a_\nu|^2 \leqq \epsilon^2$$

belong to V_n. In particular, the point

$$(\rho a_2, \rho^2 a_3, \cdots, \rho^{n-1} a_n)$$

belongs to V_n for some $\rho > 1$, and so there is a function

$$g(z) = \sum_{\nu=1}^{\infty} b_\nu z^\nu$$

of class $\mathfrak{S}$ with

$$b_\nu = a_\nu \rho^{\nu-1}, \qquad \nu = 2, 3, \cdots, n.$$

Clearly the function

$$\rho g\left(\frac{z}{\rho}\right) = \sum_{\nu=1}^{\infty} b_\nu \rho^{-\nu+1} z^\nu$$

belongs to class $\mathfrak{S}$ and it is bounded for $|z| \leqq 1$. Moreover, its coefficients are

$$b_\nu \rho^{-\nu+1} = a_\nu, \qquad \nu = 2, 3, \cdots, n,$$

and so it belongs to the point $(a_2, a_3, \cdots, a_n)$. It has thus been shown that (i) implies (ii). It is plain that (ii) implies (iii).

Assume next that (iii) is true. Let w_0 be an exterior point of the map of $|z| < 1$ by $w = f(z)$. Then there is a $\delta > 0$ such that the circle $|w - w_0| \leqq \delta$ lies