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A. A. Gonchar V. P. Havin N. K. Nikolski (Eds.)

Complex Analysis I

Entire and Meromorphic Functions, Polyanalytic Functions and Their Generalizations

复分析 I

整函数与亚纯函数，多解析函数及其广义性

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

BORIS YAKOVLEVICH LEVIN

Professor Boris Yakovlevich Levin (22 December 1906–24 August 1993) did not live to see the English edition of this work appear. His works in the theory of entire functions are considered to be classics, and are reflected in much of this survey; we are unable to convey here the impact of his works in other areas of mathematics.
We were greatly influenced by Professor B. Ya. Levin throughout our scientific careers and in particular when working on this text. We dedicate it to the memory of our teacher and friend, B. Ya. Levin, whose passing leaves a great void in the mathematical community.

A. A. Gol'dberg   I. V. Ostrovskii

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A.A. Gol'dberg, B.Ya. Levin, I.V. Ostrovskii

Translated from the Russian
by V.I. Rublinetskij and V.A. Tkachenko

Contents

Introduction

The works by Weierstrass, Mittag-Leffler and Picard dated back to the seventies of the last century marked the beginning of systematic studies of the theory of entire and meromorphic[1] functions. The theorems by Weierstrass and Mittag-Leffler gave a general description of the structure of entire and meromorphic functions. The representation of entire functions as an infinite product by Weierstrass served as the basis for studying properties of entire and meromorphic functions. The Picard theorem initiated the theory of value distribution of meromorphic functions. In 1899 Jensen proved a formula which relates the number of zeros of an entire function in a disk with the magnitude of its modulus on the circle. The Jensen formula was of a great importance for the development of the theory of entire and meromorphic functions.

The theory of entire functions was shaped as a separate scientific discipline by Laguerre, Hadamard and Borel in 1882–1900. Borel's book "Leçons sur les fonctions entières" published in 1900 was the first monograph devoted to this theory. The works by R. Nevanlinna during 1920's resulted in the intensive development of the theory of value distribution of meromorphic functions, and were largely responsible for determining its modern character. The fundamentals of this theory were presented in R. Nevanlinna's book "Le théorème de Picard-Borel et la théorie des fonctions méromorphes" (1929).

The first results in the general theory of entire functions were connected with studies of differential equations (Poincaré) and with the theory of numbers (Hadamard). In the course of further development of the theory of meromorphic and entire functions more and more links were revealed with the above-mentioned and other mathematical disciplines, such as functional analysis, mathematical physics, probability theory, etc. In the present work the authors have tried not only to give a picture of the modern state of the theory of meromorphic and entire functions, but also, to the best of their ability, to reflect the links with related disciplines.

Below follows a list of the notations which will be used hereafter without any explanations: $D_r = \{z : |z| \le r\}$; $C_r = \{z : |z| = r\}$; $D_1 = D$; $C_1 = \mathbb{T}$; $W(\theta, \epsilon) = \{z : |\arg z - \theta| \le \epsilon\}$; $S(\theta) = W(\theta, 0)$; $n(r, a, E, f)$ is a number of those a-points (with account taken of multiplicities) of a function f which lie in the set $E \cap D_r$. When writing $\lim \varphi(r)$, $O(\varphi(r))$, $o(\varphi(r))$ we always mean that $r \to \infty$.

The reference of the form Ahlfors (1937) shows the name of the author and the publication date of the item included in the reference list. In a case that there are several mathematicians of the same name we add the initials of their first names, e.g., J.Whittaker (1935) and E.Whittaker (1915).

[1] By a meromorphic function we mean a function meromorphic in $\mathbb{C}$, if not otherwise stated.

In addition to the main authors this article was written with the participation of V.S. Azarin, A.E. Eremenko, and V.A. Tkachenko. We are further indebted to A.A. Kondratyuk and M.N. Sheremeta for their valuable help in writing Section 7, Chapter 2 and Section 4, Chapter 1, respectively. The main authors are responsible for the overall concept of this article as well as its final editing.

A.A. Gol'dberg, B.Ya. Levin, I.V. Ostrovskii

Chapter 1
General Theorems on the Asymptotic Behavior of Entire and Meromorphic Functions

§1. Characteristics of Asymptotic Behavior

Entire functions are a direct generalization of polynomials, but their asymptotic behavior has an incomparably greater diversity. The most important parameter characterizing properties of a polynomial is its degree. A transcendental entire function that can be expanded into an infinite power series can be viewed as a "polynomial of infinite degree", and the fact that the degree is infinite brings no additional information to the statement that an entire function is not a polynomial. That is why, to characterize the asymptotic behavior of an entire function, one must use other quantities. For an entire function f we set

$$M(r, f) = \max\{|f(z)| : |z| = r\} .$$

Since, according to the maximum modulus principle, $M(r, f) = \max\{|f(z)| : |z| \le r\}$, then $M(r, f)$ is a non-decreasing function of $r \in \mathbb{R}_+$, and if $f \not\equiv$ const, then $M(r, f)$ strictly increases, tending to $+\infty$ for $r \to \infty$. For a polynomial f of a degree n the asymptotic relation holds $\log M(r, f) \sim n \log r$. Thus $n = \lim \log M(r, f)/\log r$, i.e., the degree of a polynomial is closely related to the asymptotics of $M(r, f)$. The ratio $\log M(r, f)/\log r$ tends to ∞ for all entire transcendental functions. That is why the growth of $\log M(r, f)$ is characterized by comparing it not with $\log r$, but with faster growing functions. The most fruitful is the comparison with power functions; in this connection we shall introduce some quantities which characterize the growth of non-decreasing functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$. The quantities

$$\rho = \rho[\alpha] = \limsup \frac{\log \alpha(r)}{\log r} , \quad \lambda = \lambda[\alpha] = \liminf \frac{\log \alpha(r)}{\log r}$$

will be called the *order* and *lower order* of a function α, respectively. If $\rho < \infty$, then the quantity

$$\sigma = \sigma[\alpha] = \limsup r^{-\rho} \alpha(r)$$

is called the *type value* of the function α. If $\sigma = \infty$, $0 < \sigma < \infty$, or $\sigma = 0$, then α is said to be of a *maximal, normal or minimal type*, respectively. If $0 < \sigma \le \infty$, then $\int_1^\infty \alpha(t) t^{-\rho-1}\, dt = \infty$. For $\sigma = 0$ this integral can either converge or diverge. In this case the function α is said to belong either to the *convergence* or to the *divergence class*. Two functions α_1 and α_2 have the *same growth category* if they have equal orders, the same type (but not necessarily equal type values!) and if they simultaneously belong either to the convergence or to the divergence class. A function α_1 has a higher growth category than α_2 in three cases: (1) $\rho[\alpha_1] > \rho[\alpha_2]$, (2) $\rho[\alpha_1] = \rho[\alpha_2] < \infty$, provided that the type of α_1 is higher than that of α_2 (the convention is that the maximal type is higher than the normal, and the normal, in its turn, is higher than the minimal), (3) $\rho[\alpha_1] = \rho[\alpha_2]$, $\sigma[\alpha_1] = \sigma[\alpha_2] = 0$, provided that α_1 belongs to the divergence class while α_2 belongs to the convergence class.

For an entire function f, the order, type, type value, convergence/divergence class and growth category are, by definition, the same as those of $\log M(r, f)$. Thus $\rho[f] = \rho[\log M(r, f)]$, and so forth. We will use the following notation: $[\rho, \sigma]$ is the class of all entire functions f such that $\rho[f] \le \rho$ and if $\rho[f] = \rho$, then $\sigma[f] \le \sigma$; $[\rho, \sigma)$ is the subclass of $[\rho, \sigma]$ with the additional condition that if $\rho[f] = \rho$, then $\sigma[f] < \sigma$. We shall also denote by $[\rho, \infty]$ the class of all entire functions f with $\rho[f] \le \rho$, and by $[\rho, \infty)$ the class of all entire functions with a growth category not higher than of order ρ and normal type. Functions of the class $[1, \infty)$ will be called *entire functions of exponential type* (EFET).

Examples. A polynomial of a positive degree is of order zero and maximal type. The functions e^z, $\sin z$ are of order 1 and normal type, i.e., they are EFETs. The function $\cos \sqrt{z}$ is of order 1/2 and normal type. The function $1/\Gamma(z)$ is of order 1 and maximal type. The function $\exp z^n$ has a normal type with respect to the order n. The function $\exp\exp z$ has an infinite order. The function $E_\rho(\sigma z; \mu)$, with

$$E_\rho(z; \mu) = \sum_{n=0}^{\infty} z^n / \Gamma(\mu + n/\rho) \,, \quad \rho > 0 \,, \quad \Re\mu > 0 \,,$$

has the order ρ and the type value σ. $E_\rho(z, \mu)$ is called a *function of the Mittag-Leffler type*, so named in honour of the mathematician who first investigated it (for $\mu = 1$). A detailed study of properties of the function $E_\rho(z, \mu)$ was undertaken by M.Dzhrbashyan (1966).

In order to determine the growth category of an entire function, the function

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

is often used. The growth categories of $\log M(r, f)$ and $m(r, f)$ coincide, although their type values can differ. This follows from the inequality

$$m(r, f) \le \log^+ M(r, f) \le \{(R + r)/(R - r)\} m(R, f), \quad 0 \le r < R < \infty \,.$$

It is well-known that the implication $(\rho[\log M(r, f)] = \lambda[\log M(r, f)]) \Leftrightarrow (\rho[m(r, f)] = \lambda[m(r, f)])$ is true. In 1963, Gol'dberg showed that the existence of the limit of $r^{-\rho} \log M(r, f)$, as $r \to \infty$, does not imply the existence of the limit of $r^{-\rho} m(r, f)$ and vice versa (see Gol'dberg and Ostrovskii (1970), pp. 100-106). Let us denote by $n(r, 0, f)$ the *counting function of zeros* of an entire function f, i.e., the number of zeros in the disk D_r, with account taken of multiplicities. Azarin (1972) showed that the existence, as $r \to \infty$, of the limits of any two out of the three functions: $r^{-\rho} \log M(r, f)$, $r^{-\rho} m(r, f)$, $r^{-\rho} n(r, 0, f)$ does not imply the existence of the other limits. This generalizes both the above-mentioned result of Gol'dberg and that of Shah proved in 1939 who compared $\log M(r,f)$ with $n(r,0,f)$.

To describe the asymptotic behavior along the rays $\{z : \arg z = \theta\}$ of an entire function f of order $\rho > 0$ and normal type the function

$$h(\theta, f) = \limsup r^{-\rho} \log |f(re^{i\theta})|, \quad 0 \le \theta \le 2\pi ,$$

can be also used. This function was introduced by Phragmén and Lindelöf in 1908 and is called the *indicator*. Its principal property is the *ρ-trigonometric convexity*, which means that for any $\theta_1 \le \theta_2 \le \theta_3 \le \theta_1 + \pi/\rho$ the inequality

$$\begin{vmatrix} \sin\rho\theta_1 & \sin\rho\theta_2 & \sin\rho\theta_3 \\ \cos\rho\theta_1 & \cos\rho\theta_2 & \cos\rho\theta_3 \\ h(\theta_1, f) & h(\theta_2, f) & h(\theta_3, f) \end{vmatrix} \ge 0$$

holds. Various consequences of the ρ-trigonometric convexity are given in Levin (1980), Chap. 1, Sect. 16. V. Bernstein proved in 1936 that every 2π-periodic ρ-trigonometrically convex function ($\rho > 0$) is the indicator of some entire function of order ρ and normal type (for $\rho = 1$ this result was obtained in 1929 by Pólya).

For $\rho = 1$ the trigonometric convexity of a 2π-periodic function h has a simple geometric interpretation. This means that h is the support function of some bounded convex set on the plane. Thus, the bounded convex set with the support function $h(\theta, f)$ corresponds to an entire function f of exponential type. The set is called the *indicator diagram* of the function f.

While investigating entire functions of finite order, there arise certain difficulties in the case of the maximal or minimal types. So it is convenient to use for comparison a broader class (than power functions), which, while retaining the principal properties of power functions, makes it possible to obtain a normal type. At the turn of the century several such classes were suggested. One of the most commonly used proved to be the one described in the book by Valiron (1923). Similar classes had been introduced by Lindelöf, Valiron and others. Following Valiron we shall call a function $\rho(r)$ the *proximate order*, if it is continuously differentiable on $\mathbb{R}_+$ and: (1) $\rho(r) \to \rho$, $0 \le \rho < \infty$, (2) $\rho'(r) r \log r \to 0$, (3) for $\rho = 0$ the property $r^{\rho(r)} \uparrow \infty$ is additionally required. For a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ of finite order a proximate order $\rho(r)$ is called the *proximate order of the function* α, if $0 < \sigma^*[\alpha] < \infty$ where

$$\sigma^*[\alpha] = \limsup r^{-\rho(r)} \alpha(r) .$$

The quantity $\sigma^*[\alpha]$ is called the *type value of the function with respect to the proximate order* $\rho(r)$. It is clear that if $\rho(r)$ is a proximate order of the function α, then $\rho(r) \to \rho = \rho[\alpha]$. The most important fact validating the use of proximate orders is that for every function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ of finite order there exists an appropriate proximate order. This theorem was proved by Valiron (in a somewhat weaker form as early as in Valiron (1914), p. 213).

The classes $[\rho(r), \sigma]$, $[\rho(r), \sigma)$ and $[\rho(r), \infty)$ are introduced in the same way as for the usual order.

In 1930, Karamata introduced the notion of slowly and regularly varying functions which found their application, see Seneta (1976), in probability theory and the theory of integral transforms. A function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *slowly varying* if for every c, $c \in (0, \infty)$, the equivalence $L(cr) \sim L(r)$ is true. By the Karamata theorem $L(cr)/L(r)$ tends to 1 uniformly with respect to $c \in [a, b] \subset (0, \infty)$. A function $V : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *regularly varying* with exponent ρ if $V(r) = r^\rho L(r)$ where L is slowly varying. It is easy to verify that for a proximate order $\rho(r)$, $\rho(r) \to \rho$, the function $V(r) = r^{\rho(r)}$ is regularly varying with the exponent ρ. Conversely, one can show that if V is regularly varying with the exponent ρ and $V(r) \to \infty$, then there exists a proximate order $\rho(r) \to \rho$ such that $r^{\rho(r)} \sim V(r)$.

The function

$$h(\theta, f) = \limsup r^{-\rho(r)} \log |f(re^{i\theta})|$$

is called the *indicator of a function* f of proximate order $\rho(r)$. The indicator is a ρ-trigonometrically convex function with $\rho = \lim \rho(r) \geq 0$ ($h(\theta, f)$ being a constant for $\rho = 0$). Whatever the proximate order $\rho(r) \to \rho > 0$, any ρ-trigonometrically convex 2π-periodic function can be the indicator of an entire function of proximate order $\rho(r)$. This result,under some additional assumptions, was obtained by Levin in 1956 (see Levin (1980)); in the general case it was obtained by Logvinenko in 1972.

Attempts were undertaken to use as comparison functions other than $V(r) = r^{\rho(r)}$, in order to examine the asymptotic behavior of entire functions; in particular, of functions of infinite order. The classic results are described in the book by Blumenthal (1910). From comparatively recent results we shall mention those by Sheremeta (1967,1968). The latter suggested a flexible growth scale containing (partly or fully) scales introduced earlier (by Schönhage, Fridman and others). In constructing his scale Sheremeta did not take, as the starting point, any elementary functions (e.g., logarithm iterations) and their superpositions, but singled out classes of functions with minimal restrictions, sufficient to obtain the needed relationships. Sheremeta's generalized orders were used not only by Sheremeta himself, but by many other researchers (Balashov , Yakovleva, Bajpaj, Juneja and others). More specific, though still rather general , scales were introduced by Klingen (1968) and by Bratishchev and Korobejnik (1976). As an example, we shall give one of the results (Sheremeta (1967)) that generalizes the well-known Hadamard formulas for calculating the order and the type value of an entire function f using its Taylor coefficients. Let α, β, γ be differentiable functions on $\mathbb{R}_+$ which tend to $+\infty$ strictly monotonically, as $r \to \infty$, and $\alpha(r + o(r)) \sim \alpha(r)$, $\beta(r + o(r)) \sim \beta(r)$, $\gamma(r + o(r)) \sim \gamma(r)$. Let $0 < p < \infty$, $0 < c < \infty$, $F(r, c, p) = \gamma^{-1}\{[\beta^{-1}(c\alpha(r))]^{1/p}\}$. Assume that $(d/dx) \log F(e^x; c, p) \to 1/p$, as $x \to +\infty$, for all c, $0 < c < \infty$ (if α and γ are slowly varying functions, only $(d/dx) \log F(e^x; c, p) = O(1)$ may be required). Then

$$\limsup_{r\to\infty} \frac{\alpha\big(\log M(r,f)\big)}{\beta\big((\gamma(r))^p\big)} = \limsup_{n\to\infty} \frac{\alpha(n/p)}{\beta\{[\gamma(e^{1/p}|a_n|^{-1/n})]^p\}} .$$

When $p = 1$, $\alpha(r) = \beta(r) = \log r$, $\gamma(r) = r$, we obtain the Hadamard formula for calculating the order, and when $p = \rho$, $\alpha(r) = \beta(r) = \gamma(r) = r$, the formula for calculating the type value.

It should be noted that there exist formulas which relate the decrease of the coefficients a_n directly with $M(r,f)$. We will quote the following result by Sheremeta (1973). Let f be an entire function, (a_n) be the sequence of its Taylor coefficients, and Φ^* be the function inverse to $\log M(e^x, f)$. If

$$\limsup(\log\log r)^{-2} \log\log M(r,f) = \infty ,$$

then

$$\limsup_{n\to\infty} n\Phi^*(n)(-\log|a_n|)^{-1} = 1 .$$

The restrictions on the growth of $M(r,f)$ in this theorem cannot be weakened. Thus, a simple universal formula is given which is applicable to all entire functions except a subclass of functions of zero order.

A proximate order $\rho(r)$ of a function α may be chosen such that not only $\sigma^*[\alpha] = 1$, but also $r^{\rho(r)} \geq \alpha(r)$ for $r \geq r_0$, and for some sequence $r_n \uparrow \infty$ the relation $r_n^{\rho(r_n)} = \alpha(r_n)$ would hold. Since $L(r) = r^{\rho(r)-\rho}$ is slowly varying, one can choose sequences $a_n \uparrow \infty$ and $\delta_n \downarrow 0$ such that $L(r)/L(r_n) \leq 1 + \delta_n$ for $r_n/a_n \leq r \leq r_n a_n$. Then $\alpha(r) \leq r^{\rho(r)} = r_n^{\rho} L(r_n)(r/r_n)^{\rho}(L(r)/L(r_n)) \leq \alpha(r_n)(r/r_n)^{\rho}(1+\delta_n)$ on this interval. It is often sufficient to use this inequality satisfied only on some sequence of intervals. Here it is unimportant that the exponent of the power majorant equals $\rho = \rho[\alpha]$, the global characteristic of the growth of the function α. This leads to the following definition: a sequence (r_n), $r_n \to \infty$, is said to be a sequence of *Pólya peaks* of order $p \in \mathbb{R}_+$ for a function α if there exist $a_n \uparrow \infty$, $\delta_n \downarrow 0$ such that

$$\alpha(r) \leq \alpha(r_n)(r/r_n)^p(1+\delta_n), \quad r_n/a_n \leq r \leq r_n a_n . \tag{1}$$

The exact upper and lower boundaries of Pólya peaks orders for a given function α will be called the *Pólya order* and the *Pólya lower order* of α, and be denoted by ρ_* and λ_*, respectively. Edrei (1965) proved that the set of Pólya peaks orders covers the interval $[\lambda, \rho]$ where λ and ρ are the lower order and order of the function α, respectively. Thus, $\lambda_* \leq \lambda \leq \rho \leq \rho_*$. It was he who introduced the term "Pólya peaks", since this notion (though implicitly and in a weaker form) was used by Pólya in 1923 in his study of the structure of infinite sequences. The set of Pólya peaks orders is the interval $[\lambda_*, \rho_*]$ for $\rho_* < \infty$ or the interval $[\lambda_*, \infty)$ for $\rho_* = \infty$ (Drasin and Shea (1972)). The Pólya order and the lower order can be determined using the formulas (Drasin and Shea (1972)):

$$\rho_* = \sup\{p : \limsup_{x,A\to\infty} \alpha(Ax)/\big(A^p\alpha(x)\big) = \infty\} ,$$