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Eberhard Freitag

Complex Analysis 2

Riemann Surfaces, Several Complex Variables, Abelian Functions, Higher Modular Functions

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Introduction

This book is intended for readers who are familiar with the basics of elementary complex function theory, i.e. the topics that are usually covered in an introductory course on the theory of complex functions. Additionally, it would be useful for the reader to be familiar with the theory of elliptic functions and the theory of elliptic modular functions. These theories were treated in detail in the textbook Complex Analysis by Rolf Busam and the present author ([FB] in the reference list of the present book). There will be many cross-references to that book.

The goal of this book is to outline the new epoch of *classical complex analysis,* which was shaped decisively by Riemann. More than a half of this volume, Chaps. I–IV, is devoted to the theory of *Riemann surfaces*.

The theory of Riemann surfaces provides a new foundation for complex analysis on a higher level. As in elementary complex analysis, the subject matter is analytic functions. But the notion of an analytic function will have now a broader meaning. The domains of definition are not exclusively open parts of the complex plane or the Riemann sphere, but more general surfaces. Such functions automatically come up when one wants to describe an a priori multivalued function such as $f(z) = \sqrt{z^4+1}$ completely by a single-valued function. The natural domain of definition of this function f will turn out to be a twofold covering of the sphere which has the shape of a torus.

This example shows in outline that, in the theory of Riemann surfaces, we have to struggle with topological problems. The notion of "topology" here has a double meaning.

First, in the present-day mathematical world, topology is a *universal linguistic* tool for addressing questions of convergence in a context that is as broad as possible. This purpose is served by the notion of a topological space and derived notions such as "open set", "closed set", "neighborhood", "continuity", "convergence", and "compactness", just to give a few important examples, similarly to *set theory*, which is also a universally valid linguistic tool in mathematics. Readers of the first volume of our book have probably gained more mathematical experience in the meantime, so we can assume that they are familiar with the language of topological spaces. For the sake of completeness, we nevertheless introduce the fundamental terms of this language in an introductory section (I.0). This contains, very briefly, all of what we need. Most of the simple proofs will be skipped.

A second aspect of topology is that it is a mathematical discipline for investigating nontrivial *geometric problems*. For example, it is an important geometric fact that every compact, orientable surface is homeomorphic to a sphere with p handles. The number p is a topological invariant of the surface which determines its topological type. Topological theorems of significant mathematical substance will be derived completely in this book. Besides the topological classification of compact oriented surfaces, we shall also give an outline of covering theory. In particular, the universal covering and its relation to the fundamental group will be treated.

By the way, the development of topology was related to the fact that it is advantageous in the theory of Riemann surfaces, as well as providing a linguistic tool and means to attack serious geometric problems in the theory.

One of the main achievements of the theory of Riemann surfaces was that it enabled a proof of the *Jacobi inversion theorem* and, moreover, opened a deep understanding of it. We shall give a complete proof of the inversion theorem in this volume.

In a similar way to that in which meromorphic functions with two independent periods, called elliptic functions, arise in the inversion of elliptic integrals, we shall be led to meromorphic functions of several complex variables $z_1, \dots z_p$ with $2p$ independent periods. Such functions are called *abelian functions*.

The inversion theorem is the prelude to a new mathematical development. It is necessary now to fix the notion of a meromorphic function of several complex variables. So, we are forced to establish a theory of complex functions of several variables. We can then introduce the notion of an *abelian function* and develop a theory of them which generalizes the theory of elliptic functions. One of the main results of this theory is that the field of abelian functions is finitely generated. It is an algebraic function field of transcendental degree $m \le p$. Unlike the case $p = 1$, we can have $m = p$ in the case $p > 1$ only under very restrictive conditions. The Riemann period relations must hold. These relations are satisfied for the abelian functions which arise from the inversion of abelian integrals. It is not only for this reason that the case $m = p$ is the most interesting one.

By studying the *manifold* of all lattices $L \subset \mathbb{C}$, we are led to the elliptic modular functions. In the same manner, abelian functions lead us to a theory of *modular functions of several complex variables*. In the last chapter of this book, we give an introduction to this theory, which has been kept as simple as possible but nevertheless leads to quite deep results.

Therefore this book is a continuation of [FB] on a higher level. The usual Cauchy–Weierstrass theory of complex functions corresponds to the theory of Riemann surfaces and to the foundation of some basics of the theory of complex functions of several complex variables. The theory of elliptic functions is replaced by the theory of abelian functions, and the theory of elliptic modular functions by the theory of Siegel's modular functions.

We have tried to proceed *in as elementary a way as possible,* to give complete proofs and to develop all that is needed. Even small excursions into algebra to develop the necessary algebraic tools are included.

It is a great pleasure for me to thank the co-author of the first volume, Rolf Busam, for his help with the figures and with the general foundations of the theory.

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