Spectral Theory of Large Dimensional Random Matrices and Its Applications to Wireless Communications and Finance Statistics

高维随机矩阵的谱理论及其在无线通信和金融统计中的应用

By
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内 容 简 介

本书讲述了随机矩阵谱理论的主要结果和前瞻研究，以及它在无线通信和现代金融风险理论中的应用。书中前面讲解基本知识，后面分析重要范例，全面介绍了随机矩阵谱理论在这两个领域中的成果。本书对其他需要高维数据分析的领域，能起到示范作用。本书可作为统计学、计算机科学、现代物理、量子力学、无线通信、金融工程、经济学等领域本科生、研究生和工程技术人员学习随机矩阵理论的重要参考资料。

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编　委　会

总　　序

侯建国
(中国科学技术大学校长、中国科学院院士、第三世界科学院院士)

大学最重要的功能是向社会输送人才. 大学对于一个国家、民族乃至世界的重要性和贡献度, 很大程度上是通过毕业生在社会各领域所取得的成就来体现的.

中国科学技术大学建校只有短短的五十年, 之所以迅速成为享有较高国际声誉的著名大学之一, 主要就是因为她培养出了一大批德才兼备的优秀毕业生. 他们志向高远、基础扎实、综合素质高、创新能力强, 在国内外科技、经济、教育等领域做出了杰出的贡献, 为中国科大赢得了"科技英才的摇篮"的美誉.

2008 年 9 月, 胡锦涛总书记为中国科大建校五十周年发来贺信, 信中称赞说: 半个世纪以来, 中国科学技术大学依托中国科学院, 按照全院办校、所系结合的方针, 弘扬红专并进、理实交融的校风, 努力推进教学和科研工作的改革创新, 为党和国家培养了一大批科技人才, 取得了一系列具有世界先进水平的原创性科技成果, 为推动我国科教事业发展和社会主义现代化建设做出了重要贡献.

据统计, 中国科大迄今已毕业的 5 万人中, 已有 42 人当选中国科学院和中国工程院院士, 是同期 (自 1963 年以来) 毕业生中当选院士数最多的高校之一. 其中, 本科毕业生中平均每 1000 人就产生 1 名院士和七百多名硕士、博士, 比例位居全国高校之首. 还有众多的中青年才俊成为我国科技、企业、教育等领域的领军人物和骨干. 在历年评选的"中国青年五四奖章"获得者中, 作为科技界、科技创新型企业界青年才俊代表, 科大毕业生已连续多年榜上有名, 获奖总人数位居全国高校前列. 鲜为人知的是, 有数千名优秀毕业生踏上国防战线, 为科技强军做出了重要贡献, 涌现出二十多名科技将军和一大批国防科技中坚.

为反映中国科大五十年来人才培养成果, 展示毕业生在科学研究中的最新进展, 学校决定在建校五十周年之际, 编辑出版《中国科学技术大学校友文库》, 于 2008 年 9 月起陆续出书, 校庆年内集中出版 50 种. 该《文库》选题经过多轮严格

的评审和论证，入选书稿学术水平高，已列为“十一五”国家重点图书出版规划.

入选作者中，有北京初创时期的毕业生，也有意气风发的少年班毕业生；有“两院”院士，也有 IEEE Fellow；有海内外科研院所、大专院校的教授，也有金融、IT 行业的英才；有默默奉献、矢志报国的科技将军，也有在国际前沿奋力拼搏的科研将才；有“文革”后留美学者中第一位担任美国大学系主任的青年教授，也有首批获得新中国博士学位的中年学者 …… 在母校五十周年华诞之际，他们通过著书立说的独特方式，向母校献礼，其深情厚意，令人感佩！

近年来，学校组织了一系列关于中国科大办学成就、经验、理念和优良传统的总结与讨论. 通过总结与讨论，我们更清醒地认识到，中国科大这所新中国亲手创办的新型理工科大学所肩负的历史使命和责任. 我想，中国科大的创办与发展，首要的目标就是围绕国家战略需求，培养造就世界一流科学家和科技领军人才. 五十年来，我们一直遵循这一目标定位，有效地探索了科教紧密结合、培养创新人才的成功之路，取得了令人瞩目的成就，也受到社会各界的广泛赞誉.

成绩属于过去，辉煌须待开创. 在未来的发展中，我们依然要牢牢把握“育人是大学第一要务”的宗旨，在坚守优良传统的基础上，不断改革创新，提高教育教学质量，早日实现胡锦涛总书记对中国科大的期待：瞄准世界科技前沿，服务国家发展战略，创造性地做好教学和科研工作，努力办成世界一流的研究型大学，培养造就更多更好的创新人才，为夺取全面建设小康社会新胜利、开创中国特色社会主义事业新局面贡献更大力量.

是为序.

2008 年 9 月

Preface

During the past three to four decades, computer science has been developed rapidly and computing facilities have been adopted in almost every discipline. At the same time, data sets collected from experiments or natural phenomena have become larger and larger in both size and dimension. As a result, the computers need to deal with these extremely huge data sets. For example, in the biological sciences, a DNA sequence can be as long as several billions. In finance research, the number of different stocks can be as large as tens of thousands. In wireless communications, the number of users supported by each base station can be several hundreds. In image processing, the pixels of a picture may be several thousands.

On the other hand, however, in statistics, classical limit theorems have been found to be seriously inadequate in aiding in the analysis of large dimensional data. All these are challenging the applicability of classical statistics. Nowadays, an urgent need to statistics is to create new limiting theories that are applicable to large dimensional data analysis. Therefore, since last decade, the large dimensional data analysis has become a very hot topic in statistics and various disciplines where statistics is applicable.

Currently, the spectral analysis of large dimensional random matrices (simply Random Matrix Theory (RMT)) is the only systematic theory that can be applied to many problems of large dimensional data analysis. The RMT dates back to the early development of Quantum Mechanics in the 1940's and 50's. In an attempt to explain the complex organizational structure of heavy nuclei, E. Wigner, Professor of Mathematical Physics at Princeton University, argued that one should not compute energy levels from Schrödinger's equation. Instead, one should imagine the complex nuclei system as a black box described by $n \times n$ Hamiltonian matrices with elements drawn from a probability distribution with only mild constraints dictated by symmetry considerations. Under these assumptions and some mild conditions imposed on the probability measure in the space of matrices, one can find the joint probability density of the n eigenvalues. Based on this consideration, Wigner established the well-known semi-circular law. Since then, RMT has been developed into an active research area in mathematical physics and probability.

Due to the need of large dimensional data analysis, the number of researchers and publications on RMT has been growing rapidly. As an evidence, we list down the following statistics searched from Mathscinet database under keyword Random Matrix on 11 April 2008:

Table 0.1 Number of publications on RMT over every 10-year period since 1955

1955–1964	1965—1974	1975—1984	1985—1994	1995—2004	2005–04.2008
23	138	249	635	1205	493

The purpose of this monograph is to introduce the basic concepts and results of RMT and some applications to wireless communications and finance statistics. The readers of this book would be graduate students and researchers who are interested in RMT and/or its applications to their own research areas. As for the theorems in RMT, we only provide an outline of their proofs. The detailed proofs are referred to the book *Spectral analysis of large dimensional random matrices* by Bai, Z. D. and Silverstein, J. W. (2006). As for the applications to wireless communications and finance statistics, we are more emphasizing the problem formulation to illustrate how the RMT is applied to, rather than the detailed mathematical derivations and proofs.

Special thanks go to Mr. Liuzhi Yin who contributed to the book by providing editing and extensive literature review, and to Ms. Yiyang Pei for proofreading.

Changchun, China
Hefei, China
Singapore

Zhidong Bai
Zhaoben Fang
Yingchang Liang
April 2008

Contents

1 Introduction

1.1 History of RMT and Current Development

The aim of this book is to introduce main results in the spectral theory of large dimensional random matrices (RM) and its rapidly spreading applications to many applied areas. As an illustration, we briefly introduce some of its applications to wireless communications and finance statistics.

In the past three or four decades, a significant and constant advancement in the world has been in the rapid development and wide application of computer science. Computing speed and storage capability have increased rapidly. This has enabled one to collect, store and analyze data sets of huge size and very high dimension. These computational developments have had strong impact on every branch of science. For example, R. A. Fisher's resampling theory had been silent for more than three decades due to the lack of efficient random number generators, until Efron proposed his renowned bootstrap in the late 1970's; the minimum L_1-norm estimation had been ignored for centuries since it was proposed by Laplace, until Huber revived it and further extended it to robust estimation in the early 1970's. It is difficult to imagine that these advanced areas in statistics would have reached such deep stages of development if there were no such assistance from the present-day computers.

Although modern computer technology helps us in so many aspects, it also brings a new and urgent task to the statisticians. All classical limiting theorems employed in statistics are derived under the assumption that the dimension of data is fixed. However, it has been found that the large dimensionality would bring intolerable error when classical limiting theorems is employed to large dimensional statistical data analysis. Then, it is natural to ask whether there are any alternative theories that can be applied to deal with large dimensional data. The theory of random matrix (RMT) has been found as a powerful tool to deal with such problems associated with large dimensional data.

1.1.1 A Brief Review of RMT

RMT traces back to the development of quantum mechanics (QM) in the 1940's and early 1950's. In QM, the energy levels of a system are described by eigenvalues of an Hermitian operator $\boldsymbol{A}$ on a Hilbert space, called the Hamiltonian. To avoid working with an infinite dimensional operator, it is common to approximate the system by discretization, amounting to a truncation, keeping only the part of the Hilbert space that is important to the problem under consideration. Hence, the limiting behavior of large dimensional random matrices attracts special interest among those working in QM, and many laws were discovered during that time. For a more detailed review on applications of RMT in QM and other related areas, the reader is referred to the books *Random Matrices* by Mehta (1991, 2004) and Bai and Silverstein (2006).

In the 1950's in an attempt to explain the complex organizational structure of heavy nuclei, Wigner, E. P., Jones Professor of Mathematical Physics at Princeton University, put forward a heuristic theory. Wigner argued that one should not try to solve the Schrödinger's equation which governs the n strongly interacting nucleons for two reasons: firstly, it is computationally prohibitive; which perhaps remains true even today with the availability of modern high speed machines and, secondly the forces between the nucleons are not very well understood. Wigner's proposal is a pragmatic one: One should not compute from the Schrödinger's equation the energy levels, but instead imagine the complex nuclei as a black box described by $n \times n$ Hamiltonian matrices with elements drawn from a probability distribution with only mild constraint dictated by symmetry consideration.

Along with this idea, Wigner (1955, 1958) proved that the expected spectral distribution of a large dimensional Wigner matrix tends to the famous semicircular law. This work was generalized by Arnold (1967, 1971) and Grenander (1963) in various aspects. Bai and Yin (1988a) proved that the spectral distribution of a sample covariance matrix (suitably normalized) tends to the semicircular law when the dimension is relatively smaller than the sample size. Following the work of Marčenko and Pastur (1967) and Pastur (1972, 1973), the asymptotic theory of spectral analysis of large dimensional sample covariance matrices was developed by many researchers including Bai, Yin, and Krishnaiah (1986), Grenander and Silverstein (1977), Jonsson (1982), Wachter (1978), Yin (1986), and Yin and Krishnaiah (1983). Also, Bai, Yin, and Krishnaiah (1986, 1987), Silverstein (1985a), Wachter (1980), Yin (1986), and Yin and Krishnaiah (1983) investigated the limiting spectral distribution (LSD) of the multivariate F-matrix, or more generally, of products of random matrices. In the early 1980's, major contributions on the existence of LSD and their explicit forms for certain classes of random matrices were made. In recent years, research on RMT is turning toward second order limiting theorems, such as the central limit theorem for linear spectral statistics, the limiting distributions of spectral spacings and extreme eigenvalues.

1.1.2 Spectral Analysis of Large Dimensional Random Matrices

Suppose $\boldsymbol{A}_n$ is an $n \times n$ matrix with eigenvalues λ_j, $j = 1, 2, \cdots, n$. If all these eigenvalues are real, e.g., if $\boldsymbol{A}_n$ is Hermitian, we can define a one-dimensional distribution function

$$F^{\boldsymbol{A}_n}(x) = \frac{1}{n}\#\{j \leqslant n : \lambda_j \leqslant x\}, \tag{1.1.1}$$

called the empirical spectral distribution (ESD) of the matrix $\boldsymbol{A}_n$. Here $\#E$ denotes the cardinality of the set E. If the eigenvalues λ_j's are not all real, we can define a two-dimensional empirical spectral distribution of the matrix $\boldsymbol{A}_n$:

$$F^{\boldsymbol{A}_n}(x, y) = \frac{1}{n}\#\{j \leqslant n : \Re(\lambda_j) \leqslant x,\ \Im(\lambda_j) \leqslant y\}. \tag{1.1.2}$$

One of the main problems in RMT is to investigate the convergence of the sequence of empirical spectral distributions $\{F^{\boldsymbol{A}_n}\}$ for a given sequence of random matrices $\{\boldsymbol{A}_n\}$. The limit distribution F (possibly defective), which is usually nonrandom, is called the *Limiting Spectral Distribution* (LSD) of the sequence $\{\boldsymbol{A}_n\}$.

We are especially interested in sequences of random matrices with dimension (number of columns) tending to infinity, which refers to *the theory of large dimensional random matrices.*

The importance of the ESD is due to the fact that many important statistics in multivariate analysis can be expressed as functions of the ESD of some RM. We now give a few examples.

Example 1.1 *Let $\boldsymbol{A}$ be an $n \times n$ positive definite matrix. Then*

$$\det(\boldsymbol{A}) = \prod_{j=1}^{n} \lambda_j = \exp\left(n \int_0^\infty \log x F^{\boldsymbol{A}}(\mathrm{d}x)\right).$$

Example 1.2 *Let the covariance matrix of a population have the form $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_q + \sigma^2 \boldsymbol{I}$, where the dimension of $\boldsymbol{\Sigma}$ is p and the rank of $\boldsymbol{\Sigma}_q$ is $q (< p)$. Suppose $\boldsymbol{S}$ is the sample covariance matrix based on n independent and identically distributed (*iid*) samples drawn from the population. Denote the eigenvalues of $\boldsymbol{S}$ by $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p$. Then the test statistic for the hypothesis H_0 : $\mathrm{rank}(\boldsymbol{\Sigma}_q) = q$ against H_1 : $\mathrm{rank}(\boldsymbol{\Sigma}_q) > q$ is given by*

$$\begin{aligned} T &= \frac{1}{p-q} \sum_{j=q+1}^{p} \sigma_j^2 - \left(\frac{1}{p-q} \sum_{j=q+1}^{p} \sigma_j\right)^2 \\ &= \frac{p}{p-q} \int_0^{\sigma_q} x^2 F^{\boldsymbol{S}}(\mathrm{d}x) - \left(\frac{p}{p-q} \int_0^{\sigma_q} x F^{\boldsymbol{S}}(\mathrm{d}x)\right)^2. \end{aligned}$$

1.1.3 Limits of Extreme Eigenvalues

In applications of the asymptotic theorems of spectral analysis of large dimensional random matrices, two important problems arose after the LSD was found. The first is the bound on extreme eigenvalues; the second is the convergence rate of the ESD, with respect to sample size. For the first problem, the literature is extensive. The first success was due to Geman (1980), who proved that the largest eigenvalue of a sample covariance matrix converges almost surely to a limit under a growth condition on all the moments of the underlying distribution. Yin, Bai, and Krishnaiah (1988) proved the same result under the existence of the 4th order moment, and Bai, Silverstein, and Yin (1988) proved that the existence of the 4th order moment is also necessary for the existence of the limit. Bai and Yin (1988b) found the necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. By the symmetry between the largest and smallest eigenvalues of a Wigner matrix, the necessary and sufficient conditions for almost sure convergence of the smallest eigenvalue of a Wigner matrix were also found.

Comparing to almost sure convergence of the largest eigenvalue of a sample covariance matrix, a relatively harder problem is to find the limit of the smallest eigenvalue of a large dimensional sample covariance matrix. The first attempt made in Yin, Bai, and Krishnaiah (1983) proved that the almost sure limit of the smallest eigenvalue of a Wishart matrix has a positive lower bound when the ratio of dimension to the degrees of freedom is less than 1/2. Silverstein (1984) modified the work by allowing the ratio less than 1. Silverstein (1985b) further proved that with probability one, the smallest eigenvalue of a Wishart matrix tends to the lower bound of the LSD when the ratio of dimension to the degrees of freedom is less than 1. However, Silverstein's approach strongly relies on the normality assumption on the underlying distribution and thus, it cannot be extended to the general case. The most latest contribution was made in Bai and Yin (1993) where it is proved that under the existence of the fourth moment of the underlying distribution, the smallest eigenvalue (when $p \leqslant n$) or the $(p-n+1)$-th smallest eigenvalue (when $p > n$) tends to $a(y) = \sigma^2(1-\sqrt{y})^2$, where $y = \lim(p/n) \in (0, \infty)$. Compared to the case of the largest eigenvalues of a sample covariance matrix, the existence of the fourth moment seems to be necessary also for the problem of the smallest eigenvalue. However, this problem has not yet been solved.

1.1.4 Convergence Rate of ESD

The second problem, the convergence rate of the spectral distributions of large dimensional random matrices, is of practical interest, but has been open for decades. In finding the limits of both the LSD and the extreme eigenvalues of symmetric random matrices, a very useful and powerful method is the moment method which does not give any information about the rate of the convergence of the ESD to the LSD. The first success was made in Bai (1993a, b), where a Berry-Esseen type inequality of the difference of two distributions was established in terms of their Stieltjes transforms. Applying this inequality, a convergence rate for the expected ESD of a large Wigner matrix was proved to be $O(n^{-1/4})$, that for the sample covariance matrix was shown to be $O(n^{-1/4})$ if the ratio of the dimension to the degrees of freedom is apart away from one, and to be $O(n^{-5/48})$, if the ratio is close to 1.

1.1.5 Circular Law

The most perplexing problem is the so-called circular law which conjectures that the spectral distribution of a non-symmetric random matrix, after suitable normalization, tends to the uniform distribution over the unit disc in the complex plane. The difficulty lies in that two most important tools used for symmetric matrices do not apply for non-symmetric matrices. Furthermore, certain truncation and centralization techniques cannot be used. The first known result was given in Mehta (1967) and in an unpublished paper of Silverstein (1984) which was reported in Hwang (1986). They considered the case where the entries of the matrix are iid standard complex normal. Their method uses the explicit expression of the joint density of the complex eigenvalues of the random matrix which was found by Ginibre (1965). The first attempt to prove this conjecture under some general conditions was made in Girko (1984a, b). However, his proofs have puzzled many who attempt to understand, without success, Girko's arguments. Recently, Edelman (1995) found the conditional joint distribution of the complex eigenvalues of a random matrix whose entries are real normal $N(0, 1)$ when the number of its real eigenvalues is given and proved that the expected spectral distribution of the real Gaussian matrix tends to the circular law. Under the existence of $4+\varepsilon$ moment and some smooth conditions, Bai (1997) proved the strong version of the circular law.

1.1.6 Central Limit Theory (CLT) of Linear Spectral Statistics

As mentioned above, functionals of the ESD of RM's are important in multivariate inference. Indeed, a parameter θ of the population can sometimes be expressed as

$$\theta = \int f(x) \mathrm{d}F(x).$$