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Tony F. Chan   Jianhong Shen

Image Processing and Analysis

Variational, PDE, Wavelet and Stochastic Methods

图像处理与分析

变分,PDE,小波及随机方法

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

To Monica, Michelle, Ryan, and Claudia

Preface

No time in human history has ever witnessed such explosive influence and impact of image processing on modern society, sciences, and technologies. From nanotechnologies, astronomy, medicine, vision psychology, remote sensoring, security screening, and the entertainment industry to the digital communication technologies, images have helped mankind to see objects in various environments and scales, to sense and communicate distinct spatial or temporal patterns of the physical world, as well as to make optimal decisions and take right actions. Image processing and understanding are therefore turning into a critical component in contemporary sciences and technologies with many important applications.

As a branch of signal processing, image processing has traditionally been built upon the machinery of Fourier and spectral analysis. In the past few decades, there have emerged numerous novel competing methods and tools for successful image processing. They include, for example, stochastic approaches based upon Gibbs/Markov random fields and Bayesian inference theory, variational methods incorporating various geometric regularities, linear or nonlinear partial differential equations, as well as applied harmonic analysis centered around wavelets.

These diversified approaches are apparently distinct but in fact intrinsically connected. Each method excels from certain interesting angles or levels of approximations but is also inevitably subject to its limitations and applicabilities. On the other hand, at some deeper levels, they share common grounds and roots, from which more efficient *hybrid* tools or methods can be developed. This highlights the necessity of *integrating* this diversity of approaches.

The present book takes a concerted step towards this integration goal by synergistically covering all the aforementioned modern image processing approaches. We strive to reveal the few key common threads connecting all these major methodologies in contemporary image processing and analysis, as well as to highlight some emergent integration efforts that have been proven very successful and enlightening. However, we emphasize that we have made no attempt to be comprehensive in covering each subarea. In addition to the efforts of organizing the vast contemporary literature into a coherent logical structure, the present book also provides some in-depth analysis for those relatively newer areas. Since very few books have attempted this integrative approach, we hope ours will fill a need in the field.

Let u denote an observed image function, and T an image processor, which can be either deterministic or stochastic, as well as linear or nonlinear. Then a typical image processing problem can be expressed by the flow chart

$$\text{(input) } u \longrightarrow \text{(processor) } T \longrightarrow \text{(output) } F = T[u],$$

where F represents important image or visual features of interest. In the present book, we explore all three key aspects of image processing and analysis.

- Modeling: What are the suitable mathematical models for u and T? What are the fundamental principles governing the constructions of such models? What are the key features that have to be properly respected and incorporated?
- Model Analysis: Are the two models for u and T compatible? Is T stable and robust to noises or general perturbations? Does $F = T[u]$ exist, and if so, is it unique? What are the fine properties or structures of the solutions? In many applications, image processors are often formulated as inverse problem solvers, and as a result, issues like stability, existence, and uniqueness become very important.
- Computation and Simulation: How can the models be efficiently computed or simulated? Which numerical regularization techniques should be introduced to ensure stability and convergence? And how should the targeted entities be properly represented?

This view governs the structure and organization of the entire book. The first chapter briefly summarizes the emerging novel field of imaging science, as well as outlines the main tasks and topics of the book. In the next two chapters, we introduce and analyze several universal modern ways for image modeling and representation (for u), which include wavelets, random fields, level sets, etc. Based on this foundation, we then in the subsequent four chapters develop and analyze four specific and significant processing models (for T) including image denoising, image deblurring, inpainting or image interpolation, and image segmentation. Embedded within various image processing models are their computational algorithms, numerical examples, or typical applications.

As the whole spectra of image processing spread so vastly, in this book we can only focus on several most representative problems which emerge frequently from applications. In terms of computer vision and artificial intelligence, these are often loosely categorized as *low-level* vision problems. We do not intend to cover *high-level* vision problems which often involve pattern learning, identification, and representation.

We are enormously grateful to Linda Thiel, Alexa Epstein, Kathleen LeBlanc, Michelle Montgomery, David Riegelhaupt, and Sara Murphy of the SIAM Publisher for their constant encouragement and care throughout the project. It has been such a wonderful experience of planning, communication, and envisaging.

We also owe profound gratitude to the following colleagues whose published works and personal discussions have greatly influenced and shaped the contents and structures of the current book (in alphabetical order): Antonin Chambolle, Ron Coifman, Ingrid Daubechies, Rachid Deriché, Ron DeVore, David Donoho, Stu Geman, Brad Lucier, Jitendra Malik, Yves Meyer, Jean-Michel Morel, David Mumford, Stan Osher, Pietro Perona, Guillermo Sapiro, Jayant Shah, James Sethian, Harry Shum, Steve Smale, Gilbert Strang, Curt Vogel, Yingnian Wu, Alan Yuille, and Song-Chun Zhu, and the list further expands.

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We also acknowledge the tremendous benefits from the participation of numerous imaging sciences related workshops at the Institute of Pure and Applied Mathematics (IPAM) at UCLA, the Institute of Mathematics and Its Applications (IMA) at the University of Minnesota, the Institute of Mathematical Sciences (IMS) at the National University of Singapore, the Mathematical Sciences Research Institute (MSRI) at Berkeley, as well as the Center of Mathematical Sciences (CMS) at Zhejiang University, China.

Finally, this book project, like all the others in our life, is an intellectual product under numerous constraints, including our busy working schedules and many other scholastic duties. Its contents and structures as presented herein are therefore only optimal subject to such inevitable conditions. All errata and suggestions for improvements will be received gratefully by the authors.

This book is absolutely impossible without the pioneering works of numerous insightful mathematicians and computer scientists and engineers. It would be our great pleasure to see that the book can faithfully reflect many major aspects of contemporary image analysis and processing. But unintentional biases are inevitable due to the limited views and experiences of the authors, and we are happy to hear any criticisms from our dear readers.

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