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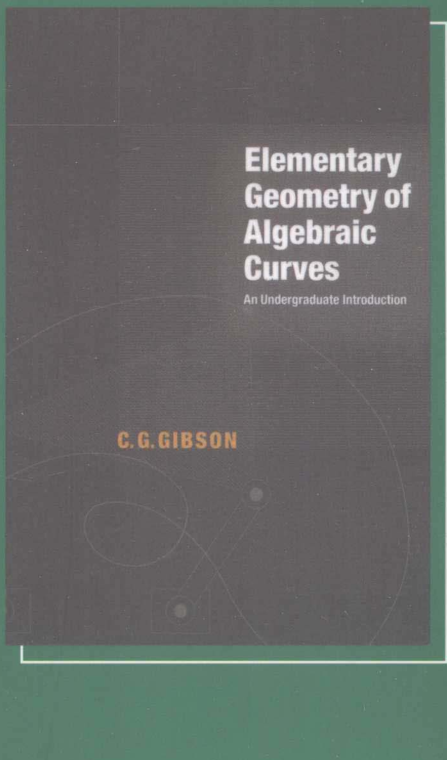
Elementary Geometry of Algebraic Curves

An Undergraduate Introduction

代数曲线几何初步

C.G.GIBSON



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C. G. Gibson

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To Dorle

Preface

For some time I have felt there is a good case for raising the profile of undergraduate geometry. The case can be argued on *academic* grounds alone. Geometry represents a way of thinking within mathematics, quite distinct from algebra and analysis, and so offers a fresh perspective on the subject. It can also be argued on purely *practical* grounds. My experience is that there is a measure of concern in various practical disciplines where geometry plays a substantial role (engineering science for instance) that their students no longer receive a basic geometric training. And thirdly, it can be argued on *psychological* grounds. Few would deny that substantial areas of mathematics fail to excite student interest: yet there are many students attracted to geometry by its sheer visual content. The decline in undergraduate geometry is a bit of a mystery. It probably has something to do with the fashion for formalism which seemed to permeate mathematics some decades ago. But things are changing. The enormous progress made in studying non-linear phenomena by geometrical methods has certainly revived interest in geometry. And for material reasons, tertiary institutions are ever more conscious of the need to offer their students more attractive courses.

0.1 General Background

I first became involved in the teaching of geometry about twenty years ago, when my department introduced an optional second year course on the geometry of plane curves, partly to redress the imbalance in the teaching of the subject. It was mildly revolutionary, since it went back to an earlier set of precepts where the differential and algebraic geometry of curves were pursued simultaneously, to their mutual advantage.

In the final year of study, students could pursue this kind of geometry by following traditional courses on the differential geometry of curves and surfaces. But in the area of algebraic geometry, matters were more problematic. A course on the geometry of algebraic curves seemed to me to be the obvious kind of development. The problem was a dearth of suitable texts. Some had developed from courses lasting for a whole session, where it was possible to attain some distance. By contrast, I was faced with a single semester course, offered over a period which saw a decline in the technical accomplishments of our students. I simply could not hope to be so ambitious. Also I find myself out of sympathy with colleagues who fret that they fail to reach significant results. I belong firmly to the school of thought which believes that it is far better to obtain a thorough appreciation of the basics than to reach some technical pinnacle. Elementary facts (for instance, the fact that the centre of a circle can be defined projectively, rather than metrically) can have a stunning impact on students. My view is that the few who wish to pursue more advanced aspects of the subject can always proceed to higher degrees where their needs will be met.

This book arose from my lecture notes after several years of experimentation. It has gained enormously from the reaction of my students over the years; they have proved to be my harshest critics, and my most helpful advisers, and I owe them a great deal.

0.2 Required Mathematical Knowledge

The intending reader will probably want to know how much mathematical knowledge is assumed of him. Let me first state quite clearly that one of my objectives was to make this book as accessible as possible. I am well aware of the needs of workers in other fields who do not have a substantial mathematical background: and I feel strongly that it is the very beginnings of the subject which need proper exposition. The more experienced reader will find that this book can be viewed as a stepping stone to many excellent texts which assume a higher level of mathematical preparation. In the area of algebra, the most basic requirement is a good understanding of the elements of linear algebra. The abstract concepts of group, domain and field do occur, and are recalled in Section 2.2: but they only occur in a fairly marginal way – you certainly should not be put off just because you are not familiar with these ideas. More substantially, much of the material rests on the unique factorization of polynomials in several variables: however, all the necessary definitions are given, the result itself is carefully stated, and references are given to the proof for those who wish to see it. In the area of analysis, I do assume the elements of calculus in several variables; basically, you need to be able to work out partial derivatives. The reader should be fluent in handling complex numbers, particularly complex roots of unity which appear in many of the calculations. Beyond that I only assume that the reader has come across the Fundamental Theorem of Algebra, i.e. the statement that every polynomial of positive degree in a single variable has at least one complex zero, but you only need the statement of the result. As to geometry, it would certainly help to have a little background (some familiarity with lines and conics for instance) but effectively the book is quite self-contained. I made a conscious decision to make the material independent of virtually any knowledge of topology. In practice that means that a small number of statements are made without proof. More regrettably, that decision precluded the possibility of developing one of the great historical ideas of the subject, that complex curves can be viewed as real surfaces.

0.3 Concerning the Structure

Concerning the structure of the book, I should say that roughly the first half is devoted to curves in the (familiar) affine plane, and the second to curves in the (less familiar) projective plane. I wanted my reader to feel quite comfortable with the mechanics of handling affine curves before making the conceptually difficult transition to the projective plane. One of the main functions of this book is to place algebraic curves in their natural setting (the complex projective plane) where their structure is more transparent. For some readers, particularly those whose background is not in mathematics, this may prove to be a psychological barrier. I can only assure such readers that the reward is much greater than the mental effort involved. History has shown that placing algebraic curves in a natural setting provides a flood of illumination, enabling one much better to comprehend the features one meets in everyday applications. I made a deliberate effort to keep the individual chapters fairly short, adopting the theory that each chapter revolves around one new idea; likewise the sections are brief, and punctuated by a series of 'examples' illustrating the concepts. I have included a collection of exercises, designed to illustrate (and even amplify) the small amount of theory. Each chapter contains sets of exercises, each appearing immediately after the relevant section. I felt it was a service to the mathematics community to gather together a coherent set of exercises for the benefit of teachers; many have been culled from the older literature.

0.4 Concerning the Content

The content of the book is largely classical. There is a tendency in the subject to overemphasize examples of curves drawn from the distant past. I wanted to make the point that the resurgence of geometry is based on the role it plays in the increasing mathematization of the physical sciences. Thus I have indulged my own passion for the curves which arise in engineering kinematics, a sadly neglected subject (the real beginnings of theoretical robotics) which deserves to be better known both for its intrinsic interest and its considerable mathematical potential. I make no apologies for the fact that conics occupy a substantial part of the text; they play a significant role in geometry at this level, and my view is that their intrinsic importance should be reflected in the space devoted to them. On the same basis, cubics receive an extended discussion. In particular, I regard the group structure on the cubic as one of the most attractive topics of elementary geometry within the reach of a mathematics undergraduate; for me, it is the mathematical equivalent of a treasured holiday snapshot. So far as objectives are concerned, I felt it was sensible to get as far as Bézout's Theorem, to justify in some measure the assertion that algebraic curves live naturally in the complex projective plane.

Some topics are conspicuous by their absence. For instance, I have great affection for the lost art of tracing algebraic curves, to which Frost's classic text on 'Curve Tracing' is a fitting memorial. Like archery, it is a satisfying pursuit, of little relevance to the world we live in. But just as the machine gun has rendered the bow obsolete, so the computer has proved itself a superbly efficient tool for tracing curves at phenomenal speeds. In this connexion, I am particularly grateful to Wendy Hawes, who constructed a picturebook of algebraic curves using the graphics facilities of the Pure Mathematics Department in The University of Liverpool, and kindly allowed me to include her pictures. Finally, I offer my warmest thanks to my friend and colleague Bill Bruce who read a working draft, and produced a wealth of valuable comment.

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Real Algebraic Curves

Plane curves arise naturally in numerous areas of the physical sciences (such as particle physics, engineering robotics and geometric optics) and within areas of pure mathematics itself (such as number theory, complex analysis and differential equations). In this introductory chapter, we will motivate some of the basic ideas and set up the underlying language of affine algebraic curves. That will also give us the opportunity to preview some of the material you will meet in the later chapters.

1.1 Parametrized and Implicit Curves

At root there are two ways in which a curve in the real plane $\mathbb{R}^2$ may be described. The distinction is quite fundamental.

- A curve may be defined *parametrically*, in the form $x = x(t)$, $y = y(t)$. The parametrization gives this image a dynamic structure: indeed at any parameter value t we have a *tangent vector* $(x'(t), y'(t))$ whose length is the *speed* of the curve at the parameter t. An example is the line parametrized by $x = t$, $y = t$, with constant speed $\sqrt{2}$, another parametrization such as $x = 2t$, $y = 2t$ yields the same image, but at twice the speed $2\sqrt{2}$.
- A curve may be defined *implicitly*, as the set of points (x, y) in the plane satisfying an equation $f(x, y) = 0$, where $f(x, y)$ is some reasonable function of x, y. For instance the line parametrized by $x = t$, $y = t$ arises from the function $f(x, y) = y - x$. Such a curve has no associated dynamic structure – it is simply a set of points in the plane.

Broadly speaking, the study of parametrized curves represents the beginnings of a major area of mathematics called *differential geometry*, whilst the study of curves defined implicitly represents the beginnings of another major area, *algebraic geometry*. It is the latter study which provides the material for this book, though at various junctures we will have something to say about the question of parametrization.

The common feature of many curves which appear in practice is that they are defined implicitly by equations of the form $f(x,y) = 0$ where $f(x,y)$ is a *real polynomial* in the variables x, y, i.e. given by a formula of the shape

$$f(x,y) = \sum_{i,j} a_{ij} x^i y^j$$

where the sum is finite and the coefficients a_{ij} are real numbers. There is much to gain in restricting attention to such curves, since they enjoy a number of important 'finiteness' properties. Moreover, it will be both profitable and illuminating to extend the concepts to situations where the coefficients a_{ij} lie in a more general 'ground field'. In some sense the complexity of a polynomial $f(x,y)$ is measured by its *degree*, i.e. the maximal value of $i + j$ over the indices i, j with $a_{ij} \neq 0$. Given a polynomial $f(x,y)$ we define its *zero set* to be

$$V_f = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}.$$

Instead of saying that a point (x,y) lies in the zero set of a curve f we may, for linguistic variety, say that (x,y) lies on the curve f, or that f passes through (x,y). Note that the zero set (and the degree) are unchanged when we multiply f by a non-zero scalar. It is for that reason that we introduce the following formal definition. A *real algebraic curve* is a non-zero real polynomial f, *up to multiplication by a non-zero scalar*. The more formally inclined reader may prefer to phrase this in terms of 'equivalence relations'. Two polynomials f, g are *equivalent*, written $f \sim g$, when there exists a non-zero scalar λ for which $g = \lambda f$. It is then trivially verified that $\sim$ has the defining properties of an equivalence relation: it is *reflexive* ($f \sim f$), it is *symmetric* (if $f \sim g$ then $g \sim f$), and it is *transitive* (if $f \sim g$ and $g \sim h$ then $f \sim h$). A real algebraic curve is then formally defined to be an equivalence class of polynomials under the relation $\sim$. So strictly speaking, a real algebraic curve is an equivalence set of all polynomials $\lambda f(x,y)$ with $\lambda \neq 0$, and any polynomial in this set is a *representative* for the curve. In this book we will usually abbreviate the term 'algebraic curve' to 'curve'. Curves of degree 1, 2, 3, 4, ... are called *lines*, *conics*, *cubics*, *quartics*, ... It is a long established convention that the curve with representative polynomial $f(x,y)$ is referred to as the 'curve' $f(x,y) = 0$. There is no harm in this provided you remember that