



Greiner

# RELATIVISTIC QUANTUM MECHANICS

WAVE EQUATIONS

Third Edition

相对论量子力学  
第3版

Springer-Verlag

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Walter Greiner

# RELATIVISTIC QUANTUM MECHANICS WAVE EQUATIONS

With a Foreword by  
D. A. Bromley

Third Edition  
With 62 Figures  
and 89 Worked Examples and Problems



Springer

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## Foreword to Earlier Series Editions

More than a generation of German-speaking students around the world have worked their way to an understanding and appreciation of the power and beauty of modern theoretical physics – with mathematics, the most fundamental of sciences – using Walter Greiner's textbooks as their guide.

The idea of developing a coherent, complete presentation of an entire field of science in a series of closely related textbooks is not a new one. Many older physicists remember with real pleasure their sense of adventure and discovery as they worked their ways through the classic series by Sommerfeld, by Planck and by Landau and Lifshitz. From the students' viewpoint, there are a great many obvious advantages to be gained through use of consistent notation, logical ordering of topics and coherence of presentation; beyond this, the complete coverage of the science provides a unique opportunity for the author to convey his personal enthusiasm and love for his subject.

The present five-volume set, *Theoretical Physics*, is in fact only that part of the complete set of textbooks developed by Greiner and his students that presents the quantum theory. I have long urged him to make the remaining volumes on classical mechanics and dynamics, on electromagnetism, on nuclear and particle physics, and on special topics available to an English-speaking audience as well, and we can hope for these companion volumes covering all of theoretical physics some time in the future.

What makes Greiner's volumes of particular value to the student and professor alike is their completeness. Greiner avoids the all too common "it follows that . . ." which conceals several pages of mathematical manipulation and confounds the student. He does not hesitate to include experimental data to illuminate or illustrate a theoretical point and these data, like the theoretical content, have been kept up to date and topical through frequent revision and expansion of the lecture notes upon which these volumes are based.

Moreover, Greiner greatly increases the value of his presentation by including something like one hundred completely worked examples in each volume. Nothing is of greater importance to the student than seeing, in detail, how the theoretical concepts and tools under study are applied to actual problems of interest to a working physicist. And, finally, Greiner adds brief biographical sketches to each chapter covering the people responsible for the development of the theoretical ideas and/or the experimental data presented. It was Auguste Comte (1798–1857) in his *Positive Philosophy* who noted, "To understand a science it is necessary to know its history". This is all too often forgotten in modern physics teaching and the

bridges that Greiner builds to the pioneering figures of our science upon whose work we build are welcome ones.

Greiner's lectures, which underlie these volumes, are internationally noted for their clarity, their completeness and for the effort that he has devoted to making physics an integral whole; his enthusiasm for his science is contagious and shines through almost every page.

These volumes represent only a part of a unique and Herculean effort to make all of theoretical physics accessible to the interested student. Beyond that, they are of enormous value to the professional physicist and to all others working with quantum phenomena. Again and again the reader will find that, after dipping into a particular volume to review a specific topic, he will end up browsing, caught up by often fascinating new insights and developments with which he had not previously been familiar.

Having used a number of Greiner's volumes in their original German in my teaching and research at Yale, I welcome these new and revised English translations and would recommend them enthusiastically to anyone searching for a coherent overview of physics.

Yale University  
New Haven, CT, USA  
1989

*D. Allan Bromley*  
Henry Ford II Professor of Physics



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## Preface to the Third Edition

We are pleased to note that our text *Relativistic Quantum Mechanics – Wave Equations* has found many friends among physics students and researchers so that the need for a third edition has arisen. We have taken this opportunity to make several amendments and improvements to the text. A number of misprints and minor errors have been corrected and explanatory remarks have been given at various places.

We thank several colleagues and students for helpful comments. We also thank Dr. Stefan Hofmann who has supervised the preparation of the third edition of the book. Finally we acknowledge the agreeable collaboration with Dr. H. J. Kölsch and his team at Springer-Verlag, Heidelberg.

Frankfurt am Main,  
March 2000

Walter Greiner

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## Preface to the Second Edition

For its second edition the book *Relativistic Quantum Mechanics – Wave Equations* has undergone only minor revisions. A number of misprints and errors in a few equations have been corrected. Also the typographical appearance and layout of the book has been improved. I hope that the book will continue to be useful to students and teachers alike.

I thank Markus Bleicher for his help and acknowledge the agreeable collaboration with Dr. H. J. Kölsch and his team at Springer-Verlag, Heidelberg.

Frankfurt am Main,  
March 1997

Walter Greiner

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## Preface to the First Edition

Theoretical physics has become a many-faceted science. For the young student it is difficult enough to cope with the overwhelming amount of new scientific material that has to be learned, let alone obtain an overview of the entire field, which ranges from mechanics through electrodynamics, quantum mechanics, field theory, nuclear and heavy-ion science, statistical mechanics, thermodynamics, and solid-state theory to elementary-particle physics. And this knowledge should be acquired in just 8–10 semesters, during which, in addition, a Diploma or Master's thesis has to be worked on or examinations prepared for. All this can be achieved only if the university teachers help to introduce the student to the new disciplines as early on as possible, in order to create interest and excitement that in turn set free essential new energy. Naturally, all inessential material must simply be eliminated.

At the Johann Wolfgang Goethe University in Frankfurt we therefore confront the student with theoretical physics immediately, in the first semester. Theoretical Mechanics I and II, Electrodynamics, and Quantum Mechanics I – An Introduction are the basic courses during the first two years. These lectures are supplemented with many mathematical explanations and much support material. After the fourth semester of studies, graduate work begins, and Quantum Mechanics II – Symmetries, Statistical Mechanics and Thermodynamics, Relativistic Quantum Mechanics, Quantum Electrodynamics, the Gauge Theory of Weak Interactions, and Quantum Chromodynamics are obligatory. Apart from these a number of supplementary courses on special topics are offered, such as Hydrodynamics, Classical Field Theory, Special and General Relativity, Many-Body Theories, Nuclear Models, Models of Elementary Particles, and Solid-State Theory. Some of them, for example the two-semester courses Theoretical Nuclear Physics and Theoretical Solid-State Physics, are also obligatory.

The form of the lectures that comprise *Relativistic Quantum Mechanics – Wave Equations* follows that of all the others: together with a broad presentation of the necessary mathematical tools, many examples and exercises are worked through. We try to offer science in as interesting a way as possible. With relativistic quantum mechanics we are dealing with a broad, yet beautiful, theme. Therefore we have had to restrict ourselves to relativistic wave equations. The selected material is perhaps unconventional, but corresponds, in our opinion, to the importance of this field in modern physics:

The Klein–Gordon equation (for spin-0 particles) and the Dirac equation (for spin- $\frac{1}{2}$  particles) and their applications constitute the backbone of these lectures. Wave equations for particles with higher spin (the Rarita–Schwinger, spin- $\frac{3}{2}$ , Kemmer and Proca, spin-1, and general Bargmann–Wigner equations) are confined to the last chapters.

After introducing the Klein–Gordon equation we discuss its properties and difficulties (especially with respect to the single-particle interpretation); the Feshbach–Villars representation is given. In many worked-out exercises and examples its practical applications can be found: pionic atoms as a modern field of research and the particularly challenging examples on the effective pion–nucleus potential (the Kisslinger potential) and its improvement by Ericson and Ericson stand in the foreground.

Most of these lectures deal with Dirac's theory. The covariance properties of the Dirac equation are discussed in detail. So, for example, its free solutions are on the one hand determined directly and on the other hand through Lorentz transformations from the simple solutions in the rest frame. Here the methodical issue is emphasized: the same physical phenomenon is illuminated from different angles. We proceed in a similar manner in the discussion of single-particle operators (the odd and even parts of an operator) and the so-called *Zitterbewegung*, which is also derived from the consideration of wave packets of plane Dirac waves. In many worked-out problems and examples the new tools are exercised. Thus the whole of Chap. 9 is dedicated to the motion of Dirac particles in external potentials. It contains simple potential problems, extensively the case of the electron in a Coulomb potential (the fine-structure formula), and muonic atoms. In Chap. 10 we present the two-centre Dirac equation, which is of importance in the modern field of heavy-ion atomic physics. The fundamental problem of overcritical fields and the decay of the electron–positron vacuum is only touched upon. A full treatment is reserved for *Quantum Electrodynamics* (Vol. 4 of this series). However, we give an extended discussion of hole theory and also of Klein's paradox. The Weyl equation for the neutrino (Chap. 14) and relativistic wave equations for particles with arbitrary spin (Chap. 15) follow. Starting with the Bargmann–Wigner equations the general frame for these equations is set, and in numerous worked-out examples and exercises special cases (spin-1 particles with and without mass, and spin- $\frac{3}{2}$  particles according to Rarita and Schwinger) are considered in greater detail. In the last chapter we give an overview of relativistic symmetry principles, which we enjoy from a superior point of view, since by now we have studied *Quantum Mechanics – Symmetries* (Vol. 2 of this series).

We hope that in this way the lectures will become ever more complete and may lead to new insights.

Biographical notes help to obtain an impression, however short, of the life and work of outstanding physicists and mathematicians. We gratefully acknowledge the publishers Harri Deutsch and F.A. Brockhaus (*Brockhaus Enzyklopädie*, F.A. Brockhaus – Wiesbaden indicated by BR) for giving permission to use relevant information from their publications.

Special thanks go to Prof. Dr. Gerhard Soff, Dr. Joachim Reinhardt, and Dr. David Vasak for their critical reading of the original draft of these lectures. Many students and collaborators have helped during the years to work out examples and exercises. For this first English edition we enjoyed the help of Maria Berenguer, Christian Borchert, Snježana Butovac, Christian Derreth, Carsten Greiner, Kordt Griepenkerl, Christian Hofmann, Raffele Mattiello, Dieter Neubauer, Jochen Rau, Wolfgang Renner, Dirk Rischke, Alexander Scherdin, Thomas Schönfeld, and

Dr. Stefan Schramm. Miss Astrid Steidl drew the graphs and prepared the figures. To all of them we express our sincere thanks.

We would especially like to thank Mr. Béla Waldhauser, Dipl.-Phys., for his overall assistance. His organizational talent and his advice in technical matters are very much appreciated.

Finally, we wish to thank Springer-Verlag; in particular, Dr. H.-U. Daniel, for his encouragement and patience, Mr. Michael Edmeades for expertly copy-editing the English edition, and Mr. R. Michels and his team for the excellent layout.

Frankfurt am Main,  
May 1990

*Walter Greiner*

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