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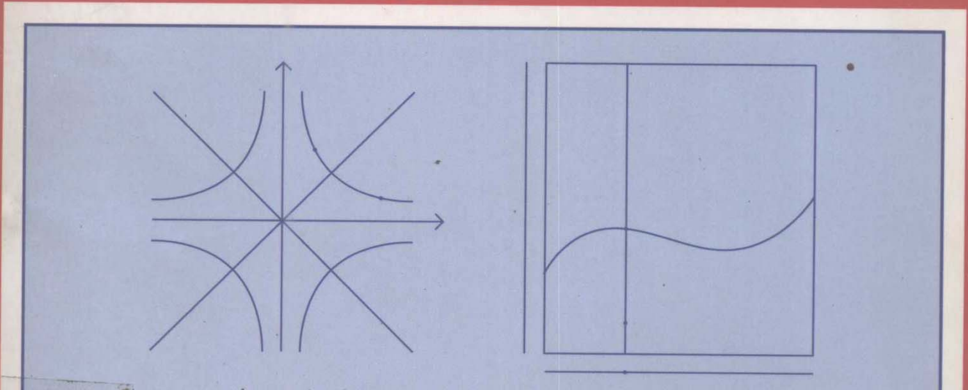
QUANTUM FIELD THEORY FOR MATHEMATICIANS

数学家用的量子场论

R. TICCIATI



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Quantum Field Theory for Mathematicians

R. TICCIATI

Maharishi University of Management

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Preface

After completing my dissertation in differential geometry, I returned to Maharishi University of Management to join the faculty there. The greatest need for my services was in the physics department, and the chairman, the well-known John Hagelin, pointed the finger of authority and said 'quantum field theory!' The class was to start in a few weeks. I laughed, but John was serious.

Fortunately, I had audited Sidney Coleman's outstanding Harvard lectures and had taken very good notes. Equally fortunate, I had Robert Brandenburger's official solutions to all the homework sets. I rolled up my sleeves and waded in.

As we battled through the material, the beautiful architecture of Coleman's course became apparent. It introduced the primary concepts – canonical quantization, renormalization, spin, functional integral quantization – one at a time and made each one practical before advancing to the next abstraction. It started with simple models and provided motivation for each elaboration.

The students, however, pinned me to the board with questions about every step in the logic. Could I produce some mathematics to fill the gap? Was there a physical principle which would justify the proposed step? The standard references failed to meet the need, and for the most part I was stumped. It was a couple of years later, when the next group of graduate students was ripening, that I found time to think out some answers. The result was a draft of the first nine chapters of this book.

Student feedback was most helpful for refining this draft, and as I began to take on the teaching of more advanced material and continued to work with successive classes, the notes developed into a pretty good book.

As this history indicates, this book is unique. The common approach in other texts is generally either that of a model builder who aims at efficient presentation of what works, or that of an applied mathematician who is trying to find a foundation for the techniques of the practicing field theorist. The model builder tends to treat technical mountains as molehills and conserves energy by omitting the details of every derivation. The applied mathematician gets absorbed in the technical mountains and does not get close to presenting anything practical. My work is half way between these two. If I may be permitted a Fullerian sentence, my text aims to provide the usual practical knowledge of QFT in comfortable stages naturally structured by the systematic introduction of fundamental principles, pleasantly enriched with overviews, summaries, and delightful mathematical details.

It has been a pleasure to work with Coleman's notes, with the many students who took my classes, and with Blue Sky Research's implementation of TEX. I would particularly like to thank Kurt Kleinschnitz for excellent suggestions, Sunil Rawal for help with the closing chapters, Geoff Golner for explaining Wilson's exact renormalization group, John Hagelin for pushing me off the deep end, and Maharishi University of Management for the financial support that made this work possible.

This book is a great satisfaction to me, and I would like to thank Cambridge University Press for making it possible to share this satisfaction with others. I hope it will be a useful reference for physics students with questions in this area and an intelligible introduction for mathematicians who are trying to find out what the physicists are up to.

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Maharishi University of Management

Introduction

This text provides a mathematical account of the principles and applied techniques of quantum field theory. It starts with the need to combine special relativity and quantum mechanics, and culminates in a basic understanding of the Standard Model of electroweak and strong interactions. The exposition is enriched with as much mathematical detail as is useful for developing a practical knowledge of quantum field theory. All the details can be brought out in a three semester course, and the main ideas can comfortably occupy two semesters. The prerequisites for this presentation are (1) familiarity with the Hilbert-space formalism of quantum mechanics, (2) assimilation of the basic principles of special relativity, and (3) a goodly measure of mathematical maturity.

The book is divided into five parts. These cover (1) canonical quantization of scalar fields, (2) Weyl, Dirac, and vector fields, (3) functional integral quantization, (4) the Standard Model of the electroweak and strong interactions, and (5) renormalization. Each of the five parts introduces one primary new principle and, having developed that principle to a goodly degree of practicality, naturally motivates its own extension in the next part.

Part 1, Chapters 1–5, presents the principles that transform quantum mechanics and special relativity into canonical perturbative quantum field theory. Its goal is to arrive at cross sections with minimal technical machinery. Part 2, Chapters 6–9, provides fields for realistic models like QED. It covers representation theory of the Lorentz group and extends the principles of Part 1 to Weyl spinors, Dirac spinors, and massive vector fields. At the end of Part 2, three weaknesses of perturbative canonical quantization are apparent, namely, its application to unstable particles, derivative interactions, and gauge theories. This motivates Part 3, Chapters 10–13, in which Haag–Ruelle scattering theory, renormalization, and functional integral quantization are developed. Using this new formalism, Part 4, Chapters 14–17, presents the structure of the Standard Model, introduces hadrons, and provides tree-level applications of the Standard Model. Finally, having taken tree-level theory about as far as is useful, Part 5 goes more deeply into regularization, renormalization, renormalizability, and the renormalization group.

The principles, examples, and techniques presented in this text are, for the most part, quite standard. However, the detailed treatment of the representation theory of the Lorentz group and $SU(3)$ in Chapters 6 and 14 respectively is unusual and helps to make this text self-sufficient. The use in Chapter 7 of Weyl spinor fields as an aid to deriving Feynman rules for Fermi fields is also unusual. The two-component spinor notation built up in that chapter is more commonly used in superfield theory.

Internal references are written without parentheses in the form '12.3.6' rather than '(12.3.6)'. To save the reader from looking up references unnecessarily, references are qualified by descriptive phrases rather than by an unhelpful noun. For example, we write 'the winding-number formula 13.7.11' rather then 'Theorem 13.7.11'.

Contents

Chapter 1

Relativistic Quantum Mechanics

Uniting the operator and state-space formalism of quantum mechanics with special relativity through a unitary representation of the Poincaré group.

Introduction

Chapters 1 to 5 constitute the first part of this book. They develop the theory of the scalar field from its roots in special relativity and quantum mechanics to its fruits in cross sections and decay rates. The technique of quantization developed here will be applied in the second part – Chapters 6 to 9 – to spinor and vector fields.

Taking quantum mechanics, with its formalism of state space, Hamiltonian, and observables, together with relativity, with its emphasis on invariance under Lorentz transformations, as the two major pillars or principles in our understanding of particle physics, the purpose of this chapter is to introduce a framework in which both principles coexist.

Section 1.1 clarifies the concept of a state space, putting physical states, position eigenstates, and momentum eigenstates into proper relationship. Section 1.2 takes the first step towards a relativistic quantum theory by promoting the energy and momentum observables into a Lorentz vector. Section 1.3 uses this vector to construct a unitary representation of translations on state space, Section 1.4 uses an independent construction to build a unitary representation of the Lorentz group, and Section 1.5 shows that these two representations determine a unitary representation of the Poincaré group. At this point, the principle of special relativity is effectively built into the formalism of quantum mechanics. Only the position operator has yet to be introduced. Section 1.6 shows that, if we adjoin eigenstates of the position operator to our state space, then there is a probability for signals to travel faster than light.

Although this probability is too small to be ruled out by experiments to date, we decide to uphold relativistic causality and drop the position operator from the theory. To model localization of events, we localize measurement using fields of observables.

For simplicity of notation, we shall choose units such that the speed of light c and the reduced Planck constant $\hbar$ are both unity:

$$c = 1 \quad \text{and} \quad \hbar = 1. \tag{1.0.1}$$

This choice of units reduces the three units time, length, and mass, to a single unit which we will generally call mass:

$$T = L = M^{-1}. \tag{1.0.2}$$

When it is necessary to compare with experiment, it will be easy to insert the factors of c and $\hbar$ which convert the units of a prediction into the units of the data. Indeed, if our result has units M^a and the physical units for the result should be $M^bL^cT^d$, then consistency requires $a = b - c - d$ and the conversion is accomplished by the equation

$$\text{(Answer in units } M^b L^c T^d) = \text{(Answer in units } M^a) \times \hbar^{b-a} \times c^{a-b-d}. \qquad (1.0.3)$$

Throughout this text, we shall take the metric on Minkowski space to be

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (1.0.4)$$

and we use the dummy suffix conventions

$$\begin{aligned} a^\mu &= g^{\mu\nu} a_\nu, \\ a \cdot b = a^\mu b_\mu &= \sum_{\mu=0}^{3} a^\mu b_\mu = g^{\mu\nu} a_\mu b_\nu, \\ a^{2k} &= (a \cdot a)^k. \end{aligned} \qquad (1.0.5)$$

1.1 One-Particle State Space – Mathematics

We want our one-particle state space to include states with definite energy and momentum. In fact we want more. We want a basis for state space in which the energy and momentum operators are simultaneously diagonal. To fulfill this desire, an excursion into mathematics is necessary. However, a little explanation will be sufficient for practical purposes. In this section, we present the little explanation and indicate the relevant mathematics in two remarks.

The function space used to model quantum-mechanical states is a Hilbert space H of square-integrable functions on physical space. The functions in H can be written either as functions of position $\bar{x}$ or as functions of momentum $\bar{p}$. The position-space representation is related to the momentum-space representation by the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(f)(\bar{p}) \stackrel{\text{def}}{=} \int dx\, e^{i\bar{x}\cdot\bar{p}} f(\bar{x}). \qquad (1.1.1)$$

Despite the change of independent variables from $\bar{x}$ to $\bar{p}$, the function $\mathcal{F}(f)$ is in H, and $\mathcal{F}$ maps H to itself.

Remark 1.1.2. Actually, if f is square-integrable, then the function $e^{i\bar{x}\cdot\bar{p}} f(\bar{x})$ is certainly square-integrable but it may not be integrable. To define the Fourier transform on H, it is necessary to define it first on some good subspace of H (such as the space of smooth functions with compact support) and then to use the fact that the Fourier transform is an isometry (up to a scale factor) on this subspace to extend the Fourier transform to an isometry of H itself.

The position operators X_r and momentum operators P_r are represented respectively by multiplication by coordinate functions x_r and the partial derivative operators $-i\partial_r$. The Fourier transform converts multiplication by x_r on functions of $\bar{x}$ into the differential operator $-i\partial_r$ on functions of $\bar{p}$:

$$\mathcal{F}(x_r f)(\bar{p}) = -i\partial_r \mathcal{F}(f)(\bar{p}). \qquad (1.1.3)$$

The eigenstates of the position operator are the delta functions:

$$(X_r\delta_{\bar{q}})(\bar{x}) = x_r\delta(\bar{x}-\bar{q}) = q_r\delta(\bar{x}-\bar{q}) = (q_r\delta_{\bar{q}})(\bar{x}), \tag{1.1.4}$$

and the eigenstates for the momentum operator P_r are the exponential functions $e_{\bar{p}}(\bar{x}) = \exp(i\bar{p}\cdot\bar{x})$:

$$(P_r e_{\bar{p}})(\bar{x}) = -i\partial_r \exp(i\bar{p}\cdot\bar{x}) = p_r \exp(i\bar{p}\cdot\bar{x}) = (p_r e_{\bar{p}})(\bar{x}). \tag{1.1.5}$$

A technical problem now arises: neither X_r nor P_r act on the whole of H, and H does not contain the eigenstates of either operator. The first part of the problem is solved by defining a subspace S of physical states in H which will be mapped into itself by X_r and P_r. The second part is solved by defining the kets (ideal states) as elements of the space S^* of continuous anti-linear functionals on S. Indeed, since P_r acts on the functions in S, these functions must be infinitely differentiable, and so S^* will contain the δ functions and all their derivatives. Similarly, taking the Fourier transform of this point, since X_r acts on S, S^* will contain the exponential functions $\exp(i\bar{p}\cdot\bar{x})$.

Thus, instead of one Hilbert space H, quantum theory uses a triple of normed vector spaces:

$$S \subset H \subset S^*. \tag{1.1.6}$$

The physical states live in S and the operator eigenstates live in S^*. With appropriate conditions on S, such a triple is called a *rigged Hilbert space.*

Remark 1.1.7. In fact, the triple $S \subset H \subset S^*$ is a rigged Hilbert space if S is a nuclear subspace of H. One condition for this is that (a) there exist a countable family $\|\ \|_k$ of norms on S with respect to which convergence is defined by

$$f_n \to f \iff \|f_n - f\|_k \to 0 \quad \text{for all } k \geq 0, \tag{1.1.8}$$

(b) S is complete with respect to this notion of convergence, and (c) there exists a Hilbert–Schmidt operator on S with a continuous inverse.

To link our example above with this structure, let H_k be the Hilbert space of functions whose kth derivatives are square-integrable, let S_k be the intersection of H_k with its Fourier transform $\mathcal{F}(H_k)$, and let $\|\ \|_k$ ($k \geq 0$) be the norm which makes S_k a Hilbert space. Then we can take S to be the intersection of the S_k. If N is the number operator of the quantum oscillator, then $1 + N$ is a Hilbert–Schmidt operator on S which has a continuous inverse.

This area of mathematics is well developed. However, its impact on field theory has been small. So we shall make no further remark on these technicalities.

In a rigged Hilbert space, the physical states have eigenstate expansions. Translating into the notation of quantum mechanics, let $|f\rangle$ be the state represented by the function f, and let $|\bar{x}\rangle$ be the position eigenstate represented by the distribution $\delta_{\bar{x}}$. We assume that the relationship between functions and kets is such that

$$f(\bar{x}) = \langle\bar{x}|f\rangle. \tag{1.1.9}$$

Then the expansion of the state $|f\rangle$ in terms of the position eigenstates $|\bar{x}\rangle$ should have the form

$$|f\rangle = \int d^3\bar{x}\, |\bar{x}\rangle\langle\bar{x}|f\rangle = \int d^3\bar{x}\, f(\bar{x})|\bar{x}\rangle. \tag{1.1.10}$$