DE GRUYTER

Vladimir A. Mikhailets,
Aleksandr A. Murach

HÖRMANDER SPACES, INTERPOLATION, AND ELLIPTIC PROBLEMS

STUDIES IN MATHEMATICS 60

DE G

Vladimir A. Mikhailets
Aleksandr A. Murach

Hörmander Spaces, Interpolation, and Elliptic Problems

De Gruyter

Mathematics Subject Classification 2000: 46E35, 46B70, 35J30, 35J40, 35J45

Authors
Prof. Dr. Vladimir Andreevich Mikhailets
National Academy of Sciences of Ukraine
Institute of Mathematics
Tereshchenkovskaya st.,3
KIEV-4
01601 UKRAINE
mikhailets@imath.kiev.ua

Prof. Dr. Aleksandr Aleksandrovich Murach
National Academy of Sciences of Ukraine
Institute of Mathematics
Tereshchenkovskaya st.,3
KIEV-4
01601 UKRAINE
murach@imath.kiev.ua

Translated by Peter V. Malyshev

ISBN 978-3-11-029685-3
e-ISBN 978-3-11-029689-1
Set-ISBN 978-3-11-029690-7
ISSN 0179-0986

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.dnb.de.

Printing and binding: CPI buch bücher.de GmbH, Birkach
♾ Printed on acid-free paper
Printed in Germany

MIX
Papier aus verantwortungsvollen Quellen
FSC www.fsc.org FSC® C003147

www.degruyter.com

De Gruyter Studies in Mathematics 60

Preface

The fundamental applications of the Sobolev spaces $W_2^m(G)$ to the investigation of many-dimensional differential equations, in particular of the elliptic type, are well known. Without the theory of spaces of this kind, the investigation of elliptic problems is, in fact, impossible. At the same time, the theory of Hörmander spaces more general than the Sobolev spaces was developed about 40 years ago. At present, there are numerous papers devoted to the applications of Hörmander spaces to differential equations.

However, the applications of Hörmander spaces to boundary-value problems for elliptic equations have been episodic up to now. The main part of the book is devoted to a fairly systematic investigation of the applications of Hörmander spaces to this class of problems. The authors introduce and study Hörmander spaces of the "intermediate" type. The functions from these spaces are characterized by the degree of smoothness intermediate between the smoothness of functions from the spaces $W_2^m(G)$ and $W_2^{m+1}(G)$, where m is an integer. As G, we can take a domain of n-dimensional Euclidean space or a compact manifold of dimension n. The first two chapters of the book are devoted to the detailed introduction and study of these spaces.

In Chapters 3 and 4, the authors consider elliptic equations and homogeneous and inhomogeneous boundary-value problems for these equations. Numerous significant results (similar to the results known for the Sobolev spaces) are obtained for these problems in Hörmander spaces. It is possible to say that the authors managed to transfer the classical "Sobolev" theory of boundary-value problems to the case of Hörmander spaces. It should also be emphasized that some problems posed independently of the notion of Hörmander spaces can be solved with the help of these spaces.

The last fifth chapter of the book is devoted to the transfer of the obtained results to the case of elliptic systems of differential equations.

I think that the book is fairly interesting and useful. It should definitely be translated into English. In this case, the results accumulated there would become accessible for a broader circle of mathematicians. In the case of translation, it would be necessary to include the proofs of various auxiliary facts mentioned in the text, which belong to the other authors. This would significantly increase the circle of possible readers of the book.

Yu. M. Berezansky,
Academician of the Ukrainian National Academy of Sciences

Preface to the English edition

The English translation of the monograph slightly differs from the Russian-language edition.

Thus, in particular, we extended the list of references, corrected the detected misprints, and improved the presentation of some results. In addition, the book is equipped with the index.

V. A. Mikhailets and A. A. Murach

Acknowledgements

The authors are especially grateful to Yu. M. Berezansky for his valuable advice and great influence, which determined, to a significant extent, their scientific interests.

We are also thankful to M. S. Agranovich, B. P. Paneyah, I. V. Skrypnik, and S. D. Eidel'man for stimulating discussions.

The support of M. L. Gorbachuk and A. M. Samoilenko, interest of B. Boyarskii, and kind participation of V. P. Burskii and S. D. Ivasyshen are also highly appreciated.

We also thank all our colleagues for their sincere interest to the new theory and its applications.

Contents

Introduction

In the theory of partial differential equations, the problems of existence, uniqueness, and regularity of solutions are in the focus of investigations. As a rule, the regularity properties of solutions are formulated in terms of belonging of these solutions to the standard classes of function spaces. Moreover, the finer the calibration of the scale of spaces, the more exact and informative the accumulated results.

Unlike the case of ordinary differential equations with smooth coefficients, these problems are fairly complicated. Indeed, some linear partial differential equations with smooth coefficients and right-hand sides are known to have no solutions in the neighborhood of a given point even in the class of distributions [113], [81, Sec. 6.0]. Moreover, some homogeneous equations (specifically, of the elliptic type) with smooth but not analytic coefficients admit nontrivial solutions with compact supports [193], [85, Theorem 13.6.5]. Therefore, the nontrivial null space of this equation cannot be removed by any homogeneous boundary conditions; i.e., the operator corresponding to any boundary-value problem for the analyzed equation is not injective. Finally, the problem of regularity of solutions is also quite complicated. Thus, even for the Laplace operator, it is known that

$$\triangle u = f \in C(\Omega) \nRightarrow u \in C^2(\Omega)$$

for any Euclidean domain Ω [64, Chap. 4, Notes].

These problems have been most completely investigated for linear elliptic equations, systems, and boundary-value problems. The fundamental results in this direction were obtained in the 1950s and 1960s by S. Agmon, A. Douglis, and L. Nirenberg [4, 5, 47], M. S. Agranovich and A. S. Dynin [12], Yu. M. Berezansky, S. G. Krein, and Ya. A. Roitberg [22, 21, 202, 203, 209], F. E. Browder [28, 29], L. R. Volevich [267, 268], J.-L. Lions and E. Magenes [121, 126], L. N. Slobodetskii [240, 241], V. A. Solonnikov [245, 246, 247], L. Hörmander [81], M. Schechter [222, 224, 225], and other researchers. In the cited works, the elliptic equations and problems were studied in the classical scales of Hölder spaces (of nonintegral order) and Sobolev spaces (both of positive and negative orders).

As a fundamental result in the theory of elliptic equations, we can mention the fact that they generate bounded Fredholm operators (i.e., operators with finite index) acting between appropriate function spaces. Thus, let $Au = f$

be a linear elliptic differential equation of order m given on a closed smooth manifold Γ. Then

$$A: H^{s+m}(\Gamma) \to H^{s}(\Gamma), \quad \text{with} \quad s \in \mathbb{R},$$

is a bounded Fredholm operator. Moreover, the finite-dimensional spaces formed by the solutions of the homogeneous equations $Au = 0$ and $A^{+}v = 0$ lie in $C^{\infty}(\Gamma)$. Here, A^{+} is the operator formally adjoint to A, whereas $H^{s+m}(\Gamma)$ and $H^{s}(\Gamma)$ are the Sobolev inner product spaces over Γ of the orders $s+m$ and s, respectively. This result implies that each solution u of the elliptic equation $Au = f$ has an important regularity property in the Sobolev scale, namely,

$$(f \in H^{s}(\Gamma) \text{ for some } s \in \mathbb{R}) \Rightarrow u \in H^{s+m}(\Gamma). \tag{1}$$

If a manifold has an edge, then the Fredholm operator is generated by an elliptic boundary-value problem for the equation $Au = f$ (e.g., by the Dirichlet boundary-value problem).

Some of these theorems were extended by H. Triebel [258, 256] and Murach [163, 164] to the scales of Nikol'skii–Besov, Zygmund, and Lizorkin–Triebel function spaces.

The cited results were applied, in various ways, to the theory of differential equations, mathematical physics, and spectral theory of differential operators (see the books by Yu. M. Berezansky [21], Yu. M. Berezansky, G. F. Us, and Z. G. Sheftel [23], O. A. Ladyzhenskaya and N. N. Ural'tseva [111], J.-L. Lions [118, 117], J.-L. Lions and E. Magenes [121], Ya. A. Roitberg [209, 210], I. V. Skrypnik [237], H. Triebel [258], the surveys by M. S. Agranovich [7, 10, 11], and references therein).

From the viewpoint of applications, especially to the spectral theory, the case of Hilbert spaces is of especial importance. Note that, until recently, the scale of Sobolev inner product spaces was the sole scale of Hilbert spaces in which the properties of elliptic operators were systematically studied. However, it was shown that the Sobolev scale is insufficiently fine for various important problems.

We present two typical examples. The first of them deals with the smoothness of the solution of the elliptic equation $Au = f$ on the manifold Γ. According to the Sobolev embedding theorem, we have

$$H^{\sigma}(\Gamma) \subset C^{r}(\Gamma) \Leftrightarrow \sigma > r + n/2, \tag{2}$$

where $r \geq 0$ is an integer and $n := \dim \Gamma$. This fact, together with property (1), allow us to study the classical smoothness of the solution u. Thus, if $f \in H^{s}(\Gamma)$ for some $s > r - m + n/2$, then $u \in H^{s+m}(\Gamma) \subset C^{r}(\Gamma)$. However, this embedding is not true for $s = r - m + n/2$; i.e., the Sobolev scale cannot be used to formulate unimprovable sufficient conditions for the inclusion $u \in C^{r}(\Gamma)$.

A similar situation is also encountered in the theory of elliptic boundary-value problems.

The second typical example corresponds to the spectral theory. We assume that a differential operator A of order $m > 0$ is elliptic on Γ and self-adjoint in the space $L_2(\Gamma)$. Consider the expansion of a function $f \in L_2(\Gamma)$ in the series

$$f = \sum_{j=1}^{\infty} c_j(f)\, h_j, \tag{3}$$

where $(h_j)_{j=1}^{\infty}$ is the complete orthonormal system of eigenfunctions of A and $c_j(f)$ are the Fourier coefficients of the function f in its expansion in h_j. The eigenfunctions are enumerated so that the moduli of the corresponding eigenvalues form a (nonstrictly) increasing sequence. By the Menchoff–Rademacher theorem (valid for general orthogonal series), expansion (3) converges almost everywhere on Γ provided that

$$\sum_{j=1}^{\infty} |c_j(f)|^2 \log^2(j+1) < \infty. \tag{4}$$

This hypotheses cannot be reformulated in equivalent way in terms of the fact that the function f belongs to Sobolev spaces because

$$\|f\|^2_{H^s(\Gamma)} \asymp \sum_{j=1}^{\infty} |c_j(f)|^2\, j^{2s}$$

for any $s > 0$. We can only state that the condition "the fact that $f \in H^s(\Gamma)$ for some $s > 0$" implies the convergence of series (3) almost everywhere on Γ. This condition does not adequately express the hypotheses (4) of the Menchoff–Rademacher theorem.

In 1963, L. Hörmander [81, Sec. 2.2] proposed a significant and useful generalization of Sobolev spaces in the category of Hilbert spaces (see also [85, Sec. 10.1]). He introduced spaces parametrized by a sufficiently general weight function playing the role of an analog of the differentiation order (or smoothness index) used for the Sobolev spaces. In particular, L. Hörmander considered the Hilbert spaces

$$B_{2,\mu}(\mathbb{R}^n) := \{u : \mu\,\mathcal{F}u \in L_2(\mathbb{R}^n)\}, \tag{5}$$

$$\|u\|_{B_{2,\mu}(\mathbb{R}^n)} := \|\mu\,\mathcal{F}u\|_{L_2(\mathbb{R}^n)},$$

where $\mathcal{F}u$ is the Fourier transform of a tempered distribution u given in $\mathbb{R}^n$, and μ is a weight function of n scalar arguments.

In the case where

$$\mu(\xi) = \langle\xi\rangle^s, \quad \langle\xi\rangle := (1 + |\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n, \quad s \in \mathbb{R},$$

we get the Sobolev space $B_{2,\mu}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ of the (differentiation) order s.

In 1965, spaces (5) were independently introduced and studied by L. R. Volevich and B. P. Paneah [269].

The Hörmander spaces occupy an especially important place among the spaces of generalized smoothness characterized by a function parameter instead of a number. These spaces serve as an object of numerous profound investigations, and a good deal of work was performed in the last decades. We refer the reader to the survey by G. A. Kalyabin and P. I. Lizorkin [90], monograph by H. Triebel [259, Sec. 22], and recent papers by A. M. Caetano and H.-G. Leopold [32], W. Farkas, N. Jacob, and R. L. Schilling [55], W. Farkas and H.-G. Leopold [56], P. Gurka, and B. Opic [70], D. D. Haroske and S. D. Moura [74, 75], S. D. Moura [162], B. Opic, and W. Trebels [177], and the references therein. Various classes of spaces of generalized smoothness naturally appear in the embedding theorems for function spaces, in the interpolation theory of function spaces, in the approximation theory, in the theory of differential and pseudodifferential operators, and in the theory of stochastic processes; see the monographs by D. D. Haroske [73], N. Jacob [87], V. G. Maz'ya and T. O. Shaposhnikova [132, Sec. 16], F. Nicola and L. Rodino [175], B. P. Paneah [181], and A. I. Stepanets [248, Chap. 1, § 7], [249, Part I, Chap. 3, Sec. 7.1], the papers by F. Cobos and D. L. Fernandez [35], C. Merucci [136], and M. Schechter [226] devoted to the interpolation of function spaces, and also the papers by D. E. Edmunds and H. Triebel [50, 51] and V. A. Mikhailets and V. Molyboga [140, 141, 142].

As early as in 1963, L. Hörmander [81] used the spaces (5) and more general Banach spaces $B_{p,\mu}(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ to study the regularity properties of the solutions of partial differential equations with constant coefficients and solutions of some classes of equations with variable coefficients given in Euclidean domains. However, unlike Sobolev spaces, the Hörmander spaces were not widely applied to the general elliptic equations on manifolds and elliptic boundary-value problems. This is explained by the long-term absence of a proper definition of Hörmander spaces on smooth manifolds (this definition should be independent of the choice of local charts covering the manifold) and the absence of analytic tools required for the effective investigation of these spaces.

For the Sobolev spaces, the required tool is available: this is the interpolation of spaces. Thus, every Sobolev space of fractional order can be obtained as a result of the interpolation of a certain pair of Sobolev spaces of integer order. This fact significantly facilitates both the investigation of these spaces and the proofs of various theorems from the theory of elliptic equations because the