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Linear Acoustic Theory.

By

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With 2 Figures.

A. Basic concepts and formulas.

Acoustical motion is, almost by definition, a perturbation. The slow compressions and expansions of materials, discussed in thermodynamics, are not thought of as acoustical phenomena, nor is the steady flow of air usually called sound. It is only when the compression is irregular enough so that over-all thermodynamic equilibrium may not be maintained, or when the steady flow is deflected by some obstacles so that wave motion is produced, that we consider part of the motion to be acoustical. In other words, we think of sound as a by-product, wanted or unwanted, of slower, more regular mechanical processes. And whether the generating process be the motion of a violin bow or the rush of gas from a turbo-jet, the part of the motion we call sound usually carries but a minute fraction of the energy present in the primary process, which is not considered to be acoustical.

This definition of acoustical motion as being the small, irregular part of some larger, more regular motion of matter, gives rise to basic difficulties when we try to develop a consistent mathematical representation of its behavior. When the irregularities are large enough, for example, there is no clear-cut way of separating the "acoustical" part from the "non-acoustical" part of the motion. For example, in the midst of turbulence, the pressure at a given point varies with time as the flow vortices move past; should this variation be called a sound wave or should it be classed as the necessary concomitant of turbulent flow? Of course this vorticity produces pressure waves which extend beyond the region of turbulence, travelling in the otherwise still fluid with the speed of sound. Here we have no trouble in deciding that this part of the motion is acoustic. For the far field the distinction between acoustic and non-acoustic motion is fairly clear; for the near field the distinction often must be arbitrary.

So any definition of the nature of sound gives rise to taxonomic difficulties. If sound is any fluid motion involving time variation of pressure, then the theory of turbulence is a branch of acoustical theory. On the other hand, if sound is that fluid disturbance which travels with the speed of sound, then not only is turbulent motion not acoustic motion but also shock waves and other, nonlinear or near-field effects are not included. In fact only in the cases where the non-steady motions are first-order perturbations of some larger, steady-state motion can one hope to make a self-consistent definition which separates acoustic from non-acoustic motion and, even here, there are ambiguities in the case of some types of near field, as will be indicated later in this article.

Thus it is not surprising that the earliest work in—and even now the majority of—acoustic theory has to do with situations where the acoustical part of the motion is small enough so that linear approximations can be used. These situations, and the linear equations which represent them, are the subject of this article. Strictly speaking, the equations to be discussed here are valid only when the acoustical component of the motion is "sufficiently" small; but it is only in this limit that we can unequivocably separate the total motion into its acoustical and its non-acoustical parts. As we have said, even here there are cases where the line of division is not completely clear, particularly when we try to represent the motion by partial differential equations and related boundary conditions; only when the representation is in terms of integral equations is the separation fairly straight-forward, even in the small-amplitude limit.

Still another limitation of the validity of acoustical theory is imposed by the atomicity of matter. The thermal motions of individual molecules, for instance, are not representable (usually) by the equations of sound; these equations are meant to represent the average behavior of large assemblies of molecules. Thus in this article, for instance, when we speak of an element of volume we implicitly assume that its dimensions, while being smaller than any wavelength of acoustical motion present, are large compared to inter-molecular spacings. Thus also, when we discuss the production of sound by the motion of boundaries, we take for granted that the boundary velocity is not greater than the mean thermal velocity of the fluid molecules.

In the present article we shall first develop the linear equations of acoustics, both their differential and integral counterparts, and discuss the various forms which are appropriate for different circumstances as well as the various basic techniques of their solution. We also discuss the effect, on these equations, of regular changes (in both time and space) of pressure, density, temperature and flow. The remainder of the article will deal with examples of the solution of these equations for situations of interest at the present time. No attempt will be made for complete coverage; the available space would preclude any such exhaustiveness, even if the authors had desired it.

I. The differential equations of linear acoustics.

1. Basic equations of motion [*1*], [*13*], [*16*], [*29*], [*60*], [*70*], [*73*]. Considering the fluid as a continuous medium, two points of view can be adopted in describing its motion. In the first, the Lagrangian description, the history of each individual fluid element, or "particle", is recorded in terms of its position as a function of time. Each particle is identified by means of a parameter, which usually is chosen to be the position vector $\boldsymbol{r}_0$ of the element at $t=0$. Thus the Lagrangian description of fluid motion is expressed by the set of functions $\boldsymbol{r}=\boldsymbol{r}(\boldsymbol{r}_0, t)$.

In the second, or Eulerian, description, on the other hand, the fluid motion is described in terms of a velocity field $\boldsymbol{V}(\boldsymbol{r}, t)$ in which $\boldsymbol{r}$ and t are now independent variables. The variation of $\boldsymbol{V}$ (or of any other fluid property, in this description) with time thus refers to a fixed point in space rather than to a specific fluid element, as in the Lagrangian description. If a field quantity is denoted by Ψ_L in the Lagrangian and by Ψ_E in the Eulerian description, the relation between the time derivatives in the two descriptions is

$$\frac{d\Psi_L}{dt} = \frac{\partial \Psi_E}{\partial t} + (\boldsymbol{V}\cdot\nabla)\,\Psi_E. \tag{1.1}$$

We note that in the case of linear acoustics for a homogeneous medium at rest we need not be concerned about the difference between $(d\Psi_L/dt)$ and $(\partial\Psi_E/\partial t)$, since the term $(\boldsymbol{V}\cdot\nabla)\,\Psi_E$ is then of second order. However in a moving or inhomogeneous medium the distinction must be maintained even in the linear approximation.

We shall ordinarily use the Eulerian description and if we ever need the Lagrangian time derivative we shall express it as $(d\Psi/dt) = (\partial\Psi/\partial t) + (\boldsymbol{V}\cdot\nabla)\Psi$ (without the subscripts). In this article the fluid motion is expressed in terms of the three velocity components V_i of the velocity vector $\boldsymbol{V}$. In addition, of course, the state of the fluid is described in terms of two independent thermodynamic variables such as pressure and temperature or density and entropy (we assume that thermodynamic equilibrium is maintained within each volume element). Thus in all we have five field variables; the three velocity components and the two independent thermodynamic variables. In order to determine these as functions of $\boldsymbol{r}$ and t we need five equations. These turn out to be the conservation laws; conservation of mass (one equation), conservation of momentum (three equations) and conservation of energy (one equation).

The *mass flow* in the fluid can be expressed by the vector components

$$\varrho V_i$$

and the *total momentum flux* by the tensor

$$t_{ij} = P_{ij} + \varrho V_i V_j,$$

in which the first term is the contribution from the thermal motion and the second terms the contribution from the gross motion of the fluid. The term P_{ij} is, of course, the *fluid stress tensor*

$$P_{ij} = (P - \varepsilon\nabla\cdot\boldsymbol{V})\,\delta_{ij} - 2\eta\,U_{ij} = P\,\delta_{ij} - D_{ij},$$

P is the total pressure in the fluid, D_{ij} is the viscous stress tensor, ε and η are two coefficients of viscosity, and

$$U_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$$

is the shear-strain tensor. In this notation the bulk viscosity would be $3\varepsilon + 2\eta$, and if this were zero (as STOKES assumed for an ideal gas) then η would equal $-3\varepsilon/2$. However acoustical measurement shows that bulk viscosity is not usually zero (in some cases it may be considerably larger than η) so we will assume that ε and η are independent parameters of the fluid.

In addition, we define the *energy density* of the fluid as

$$h = \tfrac{1}{2}\varrho V^2 + \varrho E,$$

the sum of its kinetic energy and the internal energy (E is the internal energy per unit mass) and the *energy flow vector*

$$I_i = h V_i + \sum_j P_{ij} V_j - K\frac{\partial T}{\partial x_i}$$

in which $-K(\partial T/\partial x_i)$ is the *heat flow vector*. The term $\sum_j P_{ij} V_j$ contains the work done by the pressure as well as the dissipation caused by the viscous stresses.

The basic equations of motion for the fluid, representing the conservation of mass, momentum and energy (exactly) can thus be written in the forms

$$\frac{\partial \varrho}{\partial t} + \sum_i \frac{\partial(\varrho V_i)}{\partial x_i} = Q(\boldsymbol{r}, t), \tag{1.2}$$

$$\frac{\partial(\varrho V_i)}{\partial t} + \sum_j \frac{\partial t_{ij}}{\partial x_j} = F_i(\boldsymbol{r}, t), \tag{1.3}$$

$$\frac{\partial h}{\partial t} + \sum_i \frac{\partial I_i}{\partial x_i} = H(\boldsymbol{r}, t) \tag{1.4}$$

in which Q, F_i and H are *source terms* representing the time rate of introduction of mass, momentum and heat energy into the fluid, per unit volume. The energy equation can be rewritten in a somewhat different form;

$$\varrho \frac{dE}{dt} = \varrho\left(\frac{\partial E}{\partial t} + \boldsymbol{V}\cdot\nabla E\right) = K\nabla^2 T + D - P\nabla\cdot\boldsymbol{V} + H \tag{1.5}$$

which represents the fact that a given element of fluid has its internal energy changed either by heat flow, or by viscous dissipation

$$D = \sum_{ij} D_{ij} U_{ij} = \varepsilon \sum_j U_{jj}^2 + 2\eta \sum_{ij} U_{ij}^2,$$

or by direct change of volume, or else by direct injection of heat from outside the system.

This last form of the energy equation can, of course, be obtained directly from the first law of thermodynamics $(dE/dt) = T(dS/dt) + (P/\varrho^2)(d\varrho/dt)$ if, for the rate of entropy production per unit mass, we introduce

$$T\frac{dS}{dt} = \frac{K}{\varrho}\nabla^2 T + \frac{D}{\varrho} + \frac{H}{\varrho} \tag{1.6}$$

and for the density change $d\varrho/dt$ we use $(\partial\varrho/\partial t) + \boldsymbol{V}\cdot\nabla\varrho = -\varrho\nabla\cdot\boldsymbol{V}$.

If we wish to change from one pair of thermodynamic variables to another we usually make use of the *equation of state* of the gas. For a perfect gas this is, of course,

$$P = R\varrho T \quad \text{[see also Eq. (3.1)]}. \tag{1.7}$$

2. The wave equation. Returning to Eqs. (1.2) to (1.4), by elimination of $\partial^2(\varrho V_i)/\partial x_i\,\partial t$ from the first two, we obtain

$$\frac{\partial^2\varrho}{\partial t^2} - c_0^2\nabla^2\varrho = \frac{\partial Q}{\partial t} - \sum_i \frac{\partial F_i}{\partial x_i} + \nabla^2(P - c_0^2\varrho) + \sum_{ij}\left[\frac{\partial^2 D_{ij}}{\partial x_i\,\partial x_j} + \frac{\partial^2(\varrho V_i V_j)}{\partial x_i\,\partial x_j}\right]. \tag{2.1}$$

We have here subtracted the term $c_0^2\nabla^2\varrho$ from both sides of the equation, where c_0 is the space average of the velocity of sound (c_0 can depend on t). The right-hand terms will vanish for a homogeneous, loss-less, source-free medium at rest; the result is the familiar wave equation

$$\nabla^2\varrho - \frac{1}{c_0^2}\frac{\partial^2\varrho}{\partial t^2} = 0$$

for the density. Under all other circumstances the right-hand side of Eq. (2.1) will not vanish, but will represent some sort of sound "source", either produced by external forces or injections of fluid or by inhomogeneities, motions or losses in the fluid itself (this will become more apparent when we separate the equation into its successive approximations, in the next section).

The first term, representing the injection of fluid, gives rise to a monopole wave, as will be seen in Sect. 14. For air-flow sirens and pulsed-jet engines, for example, it represents the major source term. The second term, corresponding to body forces on the fluid, gives rise to dipole waves, as will be indicated in Sect. 14. Even when this term is independent of time it may have an effect on sound transmission, as we shall see later, in the case of the force of gravity for example.

The third term on the right of Eq. (2.1) represents several effects. From Eq. (3.1) we will see that a variation of pressure is produced both by a density and an entropy variation. When the fluid changes are isentropic the term corresponds to the scattering or refraction of sound by variations in temperature or composition of the medium. It may also correspond to a source of sound, in the case of a fluctuating temperature in a turbulent medium. We shall return to this term in a later sub-section. If the motion is not isentropic, the term $\nabla^2(P - c_0^2\varrho)$ also contains contributions from entropy fluctuations in the medium. These effects will include losses produced by heat conduction and also the generation of sound by heat sources.

The fourth term, the double divergence of D_{ij}, represents the effects of viscous losses and/or the generation of sound by oscillating viscous stresses in a moving medium. If the coefficients of viscosity should vary from point to point, one would also have an effect of scattering from such inhomogeneities, but these are usually quite negligible. Finally the fifth term, the double divergence of the term $\varrho V_i V_j$ represents the scattering or the generation of sound caused by the motion of the medium [30], [31], [52]. If the two previous terms are thought of as stresses produced by thermal motion, this last term can be considered as representing the "Reynolds stress" of the gross motion; it is the major source of sound in turbulent flow. As will be indicated later, this term produces quadrupole radiation.

3. The linear approximation. After having summarised the possible effects in fluid motion we shall now consider the problem of linearisation of Eqs. (1.2) to (1.4) and the interpretation of the resulting acoustic equations. Eqs. (1.2) to (1.4) are non-linear in the variables ϱ and V_i. Not only are there terms where the product ϱV_i occurs explicitly, but also terms such as h and I_i implicitly depend on ϱ and V in a non-linear way. Furthermore, the momentum flux t_{ij} is not usually linearly related to the other field variables. In the first place the gross motion of the fluid (if there is such motion) contributes a stress $\varrho V_i V_j$ and in the second place there is a non-linear relationship between the pressure P and the other thermodynamic variables. For example, in an isentropic motion we have $(P/P_0) = (\varrho/\varrho_0)^\gamma$ and, for a non-isentropic motion we have

$$\frac{P}{P_0} = \left(\frac{\varrho}{\varrho_0}\right)^\gamma e^{(S-S_0)/C_v}. \tag{3.1}$$

Expanding this last equation, we can obtain

$$\left.\begin{aligned} P - P_0 &= \frac{\partial P_0}{\partial \varrho_0}(\varrho - \varrho_0) + \frac{\partial P_0}{\partial S_0}(S - S_0) + \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 P}{\partial \varrho_0^2}(\varrho - \varrho_0)^2 + \frac{\partial^2 P}{\partial S_0^2}(S - S_0)^2\right] + \cdots \\ &= c^2(\varrho - \varrho_0) + \frac{P_0}{C_v}(S - S_0) + \frac{1}{2}(\gamma - 1)\,c^2(\varrho - \varrho_0)^2 + \\ &\quad + \frac{P_0}{2C_v^2}(S - S_0)^2 + \cdots \end{aligned}\right\} \tag{3.2}$$

where the C's are specific heats, $\gamma = (C_p/C_v)$ and $c^2 = (\gamma P_0/\varrho_0)$. Thus only when the deviation of P from the equilibrium value P_0 is small enough is the linear relation

$$P \approx P_0 + c^2(\varrho - \varrho_0) + \frac{P_0}{C_v}(S - S_0) \tag{3.3}$$

a good approximation.

As was noted in the introduction, in acoustics we are usually concerned with the effects of some small, time-dependent deviations from the "equilibrium state" of the system. When the equilibrium state is homogeneous and static, the perturbation can easily be separated off and the resulting first-order equations are relatively simple. But when the "equilibrium state" involves inhomogeneities or steady flows the separation is less straight-forward. Even here, however, if the inhomogeneities are confined to a finite region of space, the equilibrium state outside this region being homogeneous and static, then the separating out of the acoustic motions in the outer region is not difficult. This will be discussed again in the Division on integral equations (Sects. 12 to 17).

In any case, we assume that the medium in the equilibrium state is described by the field quantities $\boldsymbol{V}_0 = \boldsymbol{v}$, P_0, ϱ_0, T_0 and S_0, for example, and define the *acoustic* velocity, pressure, density, temperature and entropy as the differences between the actual values and the equilibrium values

$$\left.\begin{aligned} \boldsymbol{u} = \boldsymbol{V} - \boldsymbol{V}_0 = \boldsymbol{V} - \boldsymbol{v}; \quad p = P - P_0; \quad \delta = \varrho - \varrho_0, \\ \theta = T - T_0; \quad \sigma = S - S_0. \end{aligned}\right\} \tag{3.4}$$

If $\boldsymbol{u}, p$ etc., are small enough we can obtain reasonably accurate equations, involving these acoustic variables to the first order, in terms of the equilibrium values (not necessarily to the first order). If we have already solved for the equilibrium state, the equilibrium values $\boldsymbol{V}_0 = \boldsymbol{v}$, P_0, etc., may be regarded as known parameters, p, $\boldsymbol{u}$, etc., being the unknowns. Thus the first order relationship between the acoustic pressure, density and entropy arising from Eq. (3.2) is

$$p \approx c^2\,\delta + \frac{P_0}{C_v}\,\sigma. \tag{3.5}$$

Our procedure will thus be to replace the quantities ϱ, $\boldsymbol{V}$, T etc., in Eqs. (1.2) to (1.5) by $(\varrho_0 + \delta)$, $(\boldsymbol{v} + \boldsymbol{u})$, $(T_0 + \theta)$, etc., and to keep only terms in first order of the acoustic quantities δ, $\boldsymbol{u}$, θ, etc. The terms containing only ϱ_0, $\boldsymbol{v}$, T_0, etc. (which we will call the *inhomogeneous terms*) need not be considered when we are computing the *propagation* of sound. On the other hand, in the study of the *generation* of sound these inhomogeneous terms are often the source terms.

In general the linear approximation thus obtained will be valid if the mean acoustic velocity amplitude $|\boldsymbol{u}|$ is small compared to the wave velocity c. There are exceptions however. In the problem of the diffraction of sound by a semi-infinite screen, for example, the acoustic velocity becomes very large in the regions close to the edge of the screen. In such regions non-linear effects are to be expected.

The linearised forms for the equations of conservation of mass, momentum and energy, and the equation of state (perfect gas), for a moving, inhomogeneous medium, are

$$\frac{\partial \delta}{\partial t} + \delta \sum_i \frac{\partial v_i}{\partial x_i} + \varrho_0 \sum_i \frac{\partial u_i}{\partial x_i} + \sum_i u_i \frac{\partial \varrho_0}{\partial x_i} \approx Q \quad (\boldsymbol{v} = \boldsymbol{V}_0), \tag{3.6}$$

$$\frac{\partial}{\partial t}(\varrho_0 u_i + \delta v_i) + \sum_j \frac{\partial}{\partial x_j}[\varrho_0(u_i v_j + u_j v_i) + \delta v_i v_j + p_{ij}] \approx F_i, \tag{3.7}$$

$$\varrho_0 T_0 \left(\frac{\partial \sigma}{\partial t} + \boldsymbol{u}\cdot\nabla S_0\right) + \frac{p}{R}\frac{dS_0}{dt} \approx K \nabla^2 \theta + 4\eta \sum_{ij} u_{ij} v_{ij} + H, \tag{3.8}$$

$$p \approx R\varrho_0 \theta + R T_0 \delta = c^2 \delta + \frac{P_0}{C_v}\sigma \tag{3.9}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\nabla);$$

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and

$$p_{ij} = p\,\delta_{ij} - d_{ij},$$
$$d_{ij} = \varepsilon \operatorname{div}(\boldsymbol{u})\,\delta_{ij} + 2\eta\, u_{ij}$$

are acoustic counterparts of the quantities defined earlier. The source terms Q, $\boldsymbol{F}$ and H are the "non-equilibrium" parts of the fluid injection, body force and heat injection; the equilibrium part of Q, for example, having been cancleed against $(\partial\varrho_0/\partial t) + \operatorname{div}(\varrho_0 \boldsymbol{v})$ from the left-hand side of (1.2).

These results are so general as to be impractical to use without further specialisation. For example, one has to assume that $\operatorname{div}\boldsymbol{v} = 0$ (usually a quite allowable assumption) before one can obtain the linear form of the general wave equation,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)^2 \delta - \nabla^2 p \approx \frac{\partial Q}{\partial t} - \nabla\cdot\boldsymbol{F} + \nabla\cdot\mathfrak{D}\cdot\nabla \tag{3.10}$$

where the last term is the double divergence of the tensor $\mathfrak{D}$, which has elements d_{ij}. In order to obtain a wave equation in terms of acoustic pressure p alone, we must determine δ and d_{ij} in terms of p. To do this in the most general case is not a particularly rewarding exercise; it is much more useful to do it for a number of specific situations which are of practical interest. This will be done in the next sections.

But, before we go to special cases, it is necessary to say a few words about the meaning of such quadratic quantities as acoustic intensity, acoustic energy density and the like. For example the energy flow vector

$$\boldsymbol{I} = (\tfrac{1}{2}\varrho V^2 + \varrho E)\,\boldsymbol{V} + \mathfrak{P}\cdot\boldsymbol{V} - K \operatorname{grad} T \tag{3.11}$$

where $\mathfrak{P}$ is the fluid stress tensor, with elements P_{ij}. The natural definition of the acoustic energy flow would be

$$\boldsymbol{i} = (\boldsymbol{I})_{\text{with sound}} - (\boldsymbol{I})_{\text{without sound}} = \boldsymbol{I} - \boldsymbol{I}_0 \tag{3.12}$$

with corresponding expressions for the acoustic energy density, $w = W - W_0$, and mass flow vector, $(\varrho\boldsymbol{V})_{\text{with sound}} - (\varrho_0 \boldsymbol{V}_0)$. Similarly with the momentum flow tensor, from which the acoustic radiation pressure tensor is obtained,

$$m_{ij} = (P_{ij} + \varrho V_i V_j)_{\text{with sound}} - (P_{ij} + \varrho V_i V_j)_{\text{without sound}}. \tag{3.13}$$

These quantities clearly will contain second order terms in the acoustic variables, therefore their rigorous calculation would require acoustic equations which are correct to the second order. As with Eq. (3.10), it is not very useful to perform this calculation in the most general case; results will be obtained later for special cases of interest. It is sufficient to point out here that the acoustic energy flow, etc., correct to second order, can indeed be expressed in terms of products of the first order acoustic variables.

In the general acoustic equations (3.6) to (3.9) we have included the source terms Q, $\boldsymbol{F}$ and H, corresponding to the rate of transfer of mass, momentum and heat energy from external sources. The sound field produced by these sources can be expressed in terms of volume integrals (see subsection 13) over these source functions. As mentioned above, we have not included terms, such as $\varrho V_i V_j$ or $\nabla^2 P_0$, which do not include the acoustic variables. The justification for this omission is that these terms balance each other locally in the equations of motion, for example fluctuations in velocities are balanced by local pressure fluctuations, and the like. These fluctuations produce sound (i.e. acoustic radiation) but in the region where the fluctuations occur (the near field) the acoustic radiation is small compared to the fluctuations themselves. However, the acoustic radiation produced by the fluctuations extends *outside* the region of fluctuation, into regions where the fluid is otherwise homogeneous and at rest (the far field) and here it can more easily be computed (and, experimentally, more easily measured).

Thus, in the study of the generation of sound by fluctuations in the fluid itself, it is essential to retain in the source terms the terms which do not contain the acoustic variables themselves. Within the region of fluctuation, the differentiation between sound and "equilibrium motion" is quite artificial (the local fluid motion could be regarded as part of the acoustic near field) and in many cases it will be more straightforward to use the original equations (1.2) to (1.5) and (2.1) in their integral form (see Sect. 13), where the net effect of the sources will appear as an integral over the region of fluctuation.

4. Acoustic equations for a fluid at rest. In this section and the next we will discuss the special forms taken on by Eqs. (3.6) to (3.13) when the equilibrium state of the fluid involves only a few of the various possible effects discussed above. At first we will assume that, in the equilibrium state, the fluid is at rest and that the acoustic changes in density are isentropic ($\sigma = 0$). In this case the relation between the acoustic pressure p and the acoustic density δ is simply

$$p = c^2 \delta; \quad c^2 = \frac{\gamma P}{\varrho} \tag{4.1}$$

from Eq. (3.3). [From here on we will omit the subscript 0 from the symbols for equilibrium values in situations like that of Eq. (4.1), where the difference between P and P_0 or ϱ and ϱ_0 would make only a second-order difference in the equations. We also will use the symbol $=$ instead of $\approx$; from now on we are committed to the linear equations.] The wave equation (3.10) then reduces to the familiar

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0. \tag{4.2}$$

Once the pressure has been computed, the other acoustic variables follow from the equations of the previous subsection;

$$\left.\begin{aligned}
&\text{Velocity,} && \boldsymbol{u} = -\frac{1}{\varrho}\int \operatorname{grad} p\, dt \\
& && \quad = \frac{1}{i k \varrho c}\operatorname{grad} p \quad (p = p_0\, e^{-i\omega t},\ \omega = k c), \\
&\text{Displacement,} && \boldsymbol{d} = \int \boldsymbol{u}\, dt = -\frac{1}{k^2 \varrho c^2}\operatorname{grad} p \quad (p = p_0\, e^{-i\omega t},\ \omega = k c), \\
&\text{Temperature,} && \theta = (\gamma - 1)\frac{T}{\varrho c^2}\, p \quad \left(\gamma = \frac{C_p}{C_v}\right), \\
&\text{Density} && \delta = \frac{p}{c^2}.
\end{aligned}\right\} \quad (4.3)$$

All these variables satisfy a homogeneous wave equation such as Eq. (4.2). For a plane sound wave, which has the general form $p = f(ct - \boldsymbol{n}\cdot\boldsymbol{r})$ (where $\boldsymbol{n}$ is a unit vector normal to the wave front), the acoustic velocity is

$$\boldsymbol{u} = \frac{\boldsymbol{n}}{\varrho c} f(ct - \boldsymbol{n}\cdot\boldsymbol{r}). \quad (4.4)$$

The quantity ϱc is called the characteristic acoustic impedance of the medium. Since div $\boldsymbol{d}$ is the relative volume change of the medium, we can use Eq. (4.1) to obtain another relation between $\boldsymbol{d}$ and p;

$$p = -\varrho c^2 \operatorname{div} \boldsymbol{d} \quad (4.5)$$

which states that the isentropic compressibility of the fluid is equal to $(1/\varrho c^2)$.

The sound energy flow vector (the *sound intensity*) is

$$\boldsymbol{i} = p\,\boldsymbol{u} = \varrho c u^2 \boldsymbol{n} = \frac{p^2}{\varrho c}\boldsymbol{n}. \quad (4.6)$$

It is tempting to consider this equation as self-evident, but it should be remembered that $\boldsymbol{i}$ is a second-order quantity which must be evaluated from Eq. (3.12). In the special case of a homogeneous medium at rest the other second-order terms cancel out and Eq. (4.6) is indeed correct to second order [*35*], [*65*]. In a moving medium, the result is not so simple [*61*].

The situation is also not so straightforward in regard to the mass flow vector. One might assume that it equals $\delta\boldsymbol{u}$, but this would result in a non-zero, time-average, mass flow for a plane wave, an erroneous result. In this case the additional second-order terms in the basic equations do contribute, making the mass flow vector zero in the second-order approximation.

On the other hand the magnitude of the acoustic momentum flux is correctly given by the expression ϱu^2 to the second order. The rate of momentum transfer is, of course, equal to the *radiation pressure* on a perfect absorber [*3*].

Generally we are interested in the time average of these quantities. For single-frequency waves (time factor $e^{-i\omega t}$) these are

$$\boldsymbol{i} = \tfrac{1}{2}\operatorname{Re}(p\,\boldsymbol{u}^*) \quad (4.7)$$

where the asterisk denotes the complex conjugate. For a plane wave [see Eq. (4.4)]

$$\boldsymbol{i} = \frac{1}{2}\varrho c|\boldsymbol{u}|^2\boldsymbol{n} = \frac{\boldsymbol{n}}{2\varrho c}|p|^2. \quad (4.8)$$

The acoustic energy density is

$$w = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{1}{2\varrho c^2}|p|^2 \quad (4.9)$$

where the first term is the kinetic energy density and the second term the potential energy density. In a plane wave these are equal. We note that the magnitude of the acoustic radiation pressure is thus equal to the acoustic energy density.

The simple wave equation (4.2) is modified when there are body forces or inhomogeneities present, even though there is no motion of the fluid in the equilibrium state, as two examples will suffice to show. For example, the force of gravity has a direct effect on the wave motion, in addition to the indirect effect produced by the change in density with height. In this case the body force $\boldsymbol{F}$ is equal to $\varrho \boldsymbol{g}$, where $\boldsymbol{g}$ is the acceleration of gravity and thus the term div $\boldsymbol{F}$ in Eq. (3.10) becomes $\boldsymbol{g} \cdot \nabla \varrho + \varrho \nabla \cdot \boldsymbol{g}$, where the magnitude of the second term is to that of the first as the wavelength is to the radius of the earth, so the second term can usually be neglected. Therefore the wave equation, in the presence of the force of gravity is

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p + \boldsymbol{g} \cdot \nabla p. \tag{4.10}$$

The added term has the effect of making the medium anisotropic. For a simple harmonic, plane wave $\exp(ik\boldsymbol{n} \cdot \boldsymbol{r} - i\omega t)$ if $\boldsymbol{n}$ is perpendicular to $\boldsymbol{g}$ then $k = (\omega/c)$, but if $\boldsymbol{n}$ is parallel to $\boldsymbol{g}$ the propagation constant k is

$$k_g = i\frac{g}{2c^2} + \frac{\omega}{c}\sqrt{1 - \frac{g^2}{4c^2\omega^2}}. \tag{4.11}$$

We note that a wave propagating downward (in the direction of $\boldsymbol{g}$) is attenuated at a rate $e^{-\alpha z}$, where $\alpha = (g/2c^2)$, independent of the frequency, and its phase velocity is $c/\sqrt{1 - (g^2/4c^2\omega^2)}$. If the frequency of the wave is less than $(g/4\pi c)$ there will be no wave motion downward.

A similar anisotropy occurs when the anisotropy is not produced by a body force but is caused by an inhomogeneity in one of the characteristics of the medium. In a solid or liquid medium the elasticity or the density may vary from point to point (as is caused by a salinity gradient in sea-water, for instance). If the medium is a gas the inhomogeneity must manifest itself by changes in temperature and/or entropy density. For a source-free medium at rest, Eq. (3.10) shows that $(\partial^2 \delta/\partial t^2) = \nabla^2 p$, but this equation reduces to the usual wave equation (4.2) only when the equilibrium entropy density is uniform and the acoustical motions are isentropic. If the equilibrium entropy density S_0 is *not* uniform the wave equation is modified, even though the acoustic motion is still isentropic.

If the acoustic disturbance is isentropic then $(dS/dt) = (\partial S/\partial t) + \boldsymbol{u} \cdot \nabla S = 0$, and if the equilibrium entropy density S_0 is a function of position but not of time, then

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{u} \cdot \nabla S_0 = 0. \tag{4.12}$$

Referring to Eqs. (3.5) and (4.3), we obtain

$$\frac{\partial \delta}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} - \frac{\varrho}{C_p}\frac{\partial \sigma}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{\varrho}{C_p}\boldsymbol{u} \cdot \nabla S_0,$$

and thus

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{C_p}\nabla p \cdot \nabla S_0$$

which, when inserted into Eq. (3.10) for a source-free medium at rest finally produces the equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 p + \frac{1}{C_p}\nabla p \cdot \nabla S_0 \tag{4.13}$$

which has the same form as Eq. (4.10) representing the effect of gravity. Thus an entropy gradient in the equilibrium state will produce anisotropy in sound propagation. As with the solutions for Eq. (4.10), sound will be attenuated in the direction of entropy increase, will be amplified in the direction of decreasing S_0. However a much larger effect arises from the fact that a change in entropy will produce a change in c from point to point, so that the coefficient of $(\partial^2 p/\partial t^2)$ in Eq. (4.13) will depend on position.

5. The effects of motion and of transport phenomena. The effects of fluid motion can be demonstrated by discussing the behavior of a non-viscous, source-free, isentropic fluid moving with uniform velocity v in the x direction, with respect to the coordinates x, y, z. In this case the wave equation (3.10) reduces to

$$\frac{\partial^2 p}{\partial t^2} + 2v\frac{\partial^2 p}{\partial x\,\partial t} + v^2\frac{\partial^2 p}{\partial x^2} - c^2 \nabla^2 p = 0. \tag{5.1}$$

The relation between the various acoustic variables may be obtained from Eqs. (3.6) to (3.9),

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\delta = -\varrho \operatorname{div} \boldsymbol{u}; \quad \varrho\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\theta = (\gamma - 1)\,T,\\ \varrho\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\boldsymbol{u} = -\operatorname{grad} p. \end{aligned}\right\} \tag{5.2}$$

It is clear that in a coordinate system moving with the medium, $x' = x - vt$, $y' = y$, $z' = z$, Eq. (5.1) reduces to the simple wave equation (4.2). However in many problems we have to do with boundaries which are at rest in the x-coordinates, so it is convenient to use Eq. (5.1). This can be simplified somewhat by changing the scale in the x direction,

$$x_1 = \frac{x}{\sqrt{1 - M^2}}; \quad M = \frac{v}{c}; \quad y_1 = y; \quad z_1 = z$$

in terms of which Eq. (5.1) becomes

$$\frac{\partial^2 p}{\partial t^2} - \frac{2Mc}{\sqrt{1 - M^2}}\frac{\partial^2 p}{\partial t\,\partial x_1} - c^2 \nabla_1^2 p = 0. \tag{5.3}$$

An alternative coordinate system, useful in studying the radiation from stationary source in a moving medium, is one moving with the medium, but contracted in the direction of motion,

$$x_2 = \frac{x'}{\sqrt{1 - M^2}} = \frac{x - vt}{\sqrt{1 - M^2}} = x_1 - \frac{vt}{\sqrt{1 - M^2}}; \quad y_2 = y; \quad z_2 = z$$

which results in the wave equation

$$\frac{1}{1 - M^2}\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial y_2^2} + \frac{\partial^2 p}{\partial z_2^2} + \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{5.4}$$

To study the effects of transport phenomena, such as viscosity and heat conduction [*22*], we consider a homogeneous, source-free medium at rest. The linearised Eqs. (3.6) to (3.9) then become

$$\left.\begin{aligned} \frac{\partial \delta}{\partial t} + \varrho \operatorname{div} \boldsymbol{u} = 0; \quad \varrho\left(\frac{\partial u_i}{\partial t}\right) + \sum_j \left(\frac{\partial p_{ij}}{\partial x_j}\right) = 0,\\ \varrho\, C_v\left(\frac{\partial \theta}{\partial t}\right) = K\nabla^2\theta - P \operatorname{div} \boldsymbol{u}; \quad p = RT\,\delta + R\varrho\,\theta \end{aligned}\right\} \tag{5.5}$$

where, as before, $p_{ij} = (p - \varepsilon \operatorname{div} \boldsymbol{u})\,\delta_{ij} - \eta\,[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$. The presence of viscosity introduces vorticity into the fluid motion, in which case it is convenient to express the acoustic velocity as the sum of an irrotational part $\boldsymbol{u}_g$ (curl $\boldsymbol{u}_g = 0$) and a rotational part $\boldsymbol{u}_v$ (div $\boldsymbol{u}_v = 0$). The equation for $\boldsymbol{u}_v$ comes from the second of Eqs. (5.5),

$$\varrho \frac{\partial \boldsymbol{u}_v}{\partial t} = \eta \nabla^2 \boldsymbol{u}_v = -\eta \operatorname{curl}(\operatorname{curl} \boldsymbol{u}_v) \qquad (5.6)$$

which corresponds to the diffusion of vorticity, caused by viscosity, into the sound field.

The irrotational part of $\boldsymbol{u}$ can most easily be obtained from the gradient of θ. By eliminating div $\boldsymbol{u}$ between the second and third of Eqs. (5.5), we can obtain a differential equation for θ. In the case of harmonic time dependence, where the time factors are $e^{-i\omega t}$ and $(\omega/c) = k$, this equation simplifies to the fourth-order form

$$(\nabla^2 + k_p^2)(\nabla^2 + k_h^2)\,\theta = 0 \qquad (5.7)$$

where

$$\left.\begin{aligned} & k_p^2 = -\frac{B}{2D}(1 - A); \quad k_h^2 = -\frac{B}{2D}(1 + A), \\ & A^2 = 1 + \frac{4\omega^2 D}{B^2}; \quad B = c^2\left[1 - \frac{2}{3} i k^2 (l_v^2 + \gamma l_h^2)\right], \\ & D = i\frac{l_h^2 c^2}{2}\left[1 - \frac{2}{3} i \gamma (k l_v)^2\right], \\ & l_v^2 = \frac{2\eta}{\omega\varrho}; \quad l_h^2 = \frac{2K}{\varrho\omega C_p}. \end{aligned}\right\} \qquad (5.8)$$

As will be shown in Sect. 11, the lengths l_v and l_h are the viscous and thermal boundary layer thicknesses. For air at normal pressure and temperature l_v and l_h are both approximately 0.007 cm at 1000 cps, quite small compared to most wavelengths. In this case, when kl_v and kl_h are small, the two propagation constants are approximately

$$\left.\begin{aligned} & k_p \approx k\left\{1 + i\left[\frac{1}{3}(k l_v)^2 + \frac{1}{4}(\gamma - 1)(k l_h)^2\right]\right\}; \\ & k_h \approx \frac{1 + i}{l_h}. \end{aligned}\right\} \qquad (5.9)$$

The solution of Eq. (5.7) is $\theta = \theta_p + \theta_h$, where the two components are solutions of the separated equations $(\nabla^2 + k_p^2)\,\theta_p = 0$ and $(\nabla^2 + k_h^2)\,\theta_h = 0$, respectively. The exact forms and magnitudes of the two components will be determined by the boundary conditions, as will be shown in the next Division (Sect. 11). Propagation constant k_p corresponds to the usual wave motion, with a small attenuation caused by both viscosity and heat conduction. Constant k_h corresponds to thermal boundary waves near a conducting surface; these waves are negligible more than a distance l_h away from the surface.

The corresponding acoustic velocity and pressure can be obtained once θ_p and θ_h have been found. For example the velocity is

$$\boldsymbol{u} = \boldsymbol{u}_v + \frac{i\omega}{T k_p^2}\left[1 - \frac{1}{2}\gamma (k_p l_h)^2\right] \operatorname{grad} \theta_p + \frac{i\omega}{T k_h^2}\left[1 - \frac{1}{2}\gamma (k_s l_h)^2\right] \operatorname{grad} \theta_h. \quad (5.10)$$

The first and last terms are important only near boundary surfaces, the first forming the viscous boundary layer and the third the thermal boundary layer. The second term represents the main contribution to the acoustic field away from these boundary layers.

6. Internal energy losses. For some purposes it may be sufficient to compute the energy lost per unit volume of the fluid, as a quadratic function of the acoustic variables, rather than to work out the solution of Eqs. (5.6) and (5.7). In this Division we consider the losses in the main body of the medium; in the next Division (Sect. 11) we treat losses near a boundary surface. We also confine our discussion to the effects of viscosity and heat conduction on a homogeneous fluid at rest.

The average rate of energy loss per unit volume of the fluid can be calculated from the time average of the "loss function"

$$L = -\sum_{ij} P_{ij} U_{ij},$$

where, from Sect. 1, P_{ij} are elements of the stress tensor and U_{ij} elements of the strain tensor. Separating

$$P_{ij} = (P_0 + p)\,\delta_{ij} - D_{ij}$$

into its pressure and viscous components, we have

$$L = -(P_0 + p)\,\nabla\cdot\boldsymbol{u} + \sum_{ij} D_{ij} U_{ij}. \tag{6.1}$$

If we consider harmonic time dependence (time factor $e^{-i\omega t}$) only and disregard second-order terms, the time average of $P_0 \nabla\cdot\boldsymbol{u}$ is zero. Since also $\nabla\cdot\boldsymbol{u} = -(1/\varrho)\,(\partial\delta/\partial t)$, we can write the average acoustic energy loss per unit volume per second as

$$\overline{L} = \left[\frac{p}{\varrho}\,\frac{\partial\delta}{\partial t}\right]_{\text{av}} + \left[\sum_{ij} D_{ij} U_{ij}\right]_{\text{av}}. \tag{6.2}$$

The first term represents the average work done on the medium by the acoustic pressure, which results in an increase of the internal energy of the medium and a "leakage" of energy due to heat conduction. The heat leakage produces a phase difference between the pressure p and the density δ so that the time average of $p(\partial\delta/\partial t)$ differs from zero.

When the time factor is $e^{-i\omega t}$ the first term in (6.2) can be expressed as $\frac{1}{2}\,\mathrm{Re}\,[-i\omega p^*\,\delta/\varrho]$ where p^* is the complex conjugate of p. Introducing the compressibility $\varkappa = (\delta/\varrho p)$ of the fluid, we obtain

$$\overline{L} = \tfrac{1}{2}\,\mathrm{Re}\,(-i\,\omega\,\varkappa\,|p|^2) + \overline{D}; \quad \overline{D} = \left[\sum_{ij} D_{ij} U_{ij}\right]_{\text{av}}. \tag{6.3}$$

The viscous dissipation function $\overline{D}$ can be calculated directly from the acoustic velocity field, and needs no further discussion here. The evaluation of the compressibility $\varkappa$, however, requires further discussion.

From the linearised equations (3.5) to (3.9) we have (when v is zero and S_0 is constant)

$$\left.\begin{aligned} \varrho\,T\,\frac{\partial\sigma}{\partial t} &= K\,\nabla^2\theta \approx \frac{K}{\varrho C_p}\,\nabla^2 p \approx -k^2\,\frac{K p}{\varrho C_p}\,; \\ p &= c^2\,\delta + \frac{P}{C_v}\,\sigma; \quad k = \frac{\omega}{c} \end{aligned}\right\} \tag{6.4}$$

where we have used $\theta = (p/\varrho C_p) + (T\sigma/C_p)$ and have discarded the term containing $K\sigma$ as being of order K^2. By eliminating σ in these two equations we obtain a first-order expression for the compressibility

$$\varkappa = \frac{1}{\varrho c^2}\left[1 + i\,\frac{\gamma - 1}{2\gamma}\,(k\,l_h)^2\right] \tag{6.5}$$