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Functional Inequatities, Markov Semigroups and Spectral Theory

（泛函不等式，马尔可夫半群与谱理论）

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Functional Inequalities, Markov Semigroups and Spectral Theory

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Preface

Since in a standard situation (e.g. in the symmetric case), any C_0-contraction semigroup (and hence its generator) on a Hilbert space is uniquely determined by the associated quadratic form, it is reasonable to describe the properties of the semigroup and its generator by using functional inequalities of the quadratic form. In particular, if the associated form is a Dirichlet form, then the corresponding semigroup is (sub-) Markovian. The purpose of this book is to present a systematic account of functional inequalities for Dirichlet forms and applications to Markov semigroups (or Markov processes in a regular case).

The functional inequalities considered here only involve in the Dirichlet form and one or two norms of functions, and can be easily illustrated in many cases. On the other hand, these inequalities imply plentiful analytic properties of Markov semigroups and generators, which are related to various behaviors of the corresponding Markov processes. For instance, the Poincaré inequality is equivalent to the exponential convergence of the semigroup and the existence of the spectral gap. Moreover, the Gross log-Sobolev inequality is equivalent to Nelson's hypercontractivity of the semigroup and is strictly stronger than the Poincaré inequality. So, it is natural for us to ask for more spectral information and semigroup properties from more general functional inequalities. This is the starting point of the book.

In this book, we introduce functional inequalities to describe:

(i) the spectrum of the generator: the essential and discrete spectrums, high order eigenvalues, the principal eigenvalue, and the spectral gap;

(ii) the semigroup properties: the uniform integrability, the compactness, the convergence rate, and the existence of density;

(iii) the reference measure and the intrinsic metric: the concentration, the isoperimetric inequality, and the transportation cost inequality.

For reader's convenience and for the completeness of the account, we summarize some necessary preliminaries in Chapter 0. Corresponding to various levels of spectral and semigroup properties, Chapters 1, 3, 4, 5 and 6 focus on several different functional inequalities respectively: Chapter 1 and Chapter 5 introduce the above mentioned Poincaré and log-Sobolev inequalities respectively, Chapter 6 the interpolations of these two inequalities, Chapter 3 the super Poincaré inequality, and Chapter 4 the weak Poincaré inequality. Each of these chapters presents a correspondence between the underlying functional inequality and the properties of the semigroup and its generator, as well as sufficient and necessary conditions for the functional inequality to hold. Moreover, the general results are illustrated by concrete examples, in particular, examples of diffusion processes on manifolds and countable Markov chains. These chapters are relatively (although not absolutely) independent, so that one may read in one's own order without much trouble.

Chapter 2 is devoted to diffusion processes on Riemannian manifolds and applications to geometry analysis. In particular, the estimation of the first eigenvalue is related to the Poincaré inequality, while the results concerning gradient estimates, the Harnack inequality and the isoperimetric inequality will be used in the sequel to illustrate other functional inequalities. The results included in §2.2 concerning the first eigenvalue have been introduced in a recent monograph [47] by Professor Mu-Fa Chen. Chen's monograph emphasizes the main idea of the study which is crucial for understanding the machinery of the work, while the present book provides the technical details which are useful for further study. Finally, in Chapter 7 we establish functional inequalities for three infinite-dimensional models which have been studying intensively in stochastic analysis and mathematical physics.

At the end of each chapter (except Chapter 0), some historical notes and open questions for further studies are addressed. The notes are not intended to summarize the principal results of each paper cited but merely to indicate the connection to the main contents of each chapter in question, while the open problems are listed mainly based on my own interests. Thus, these notes are far from complete in the strict sense. At the end of the book, a list of publications and an index of main notations and key words are presented for reader's reference. These references are presented not for completeness but for a usable guide to the literature. I regret that there might be a lot of related publications which have not been mentioned in the book.

Due to the limitation of knowledge and the experience of writing, I would like to apologize in advance for possible mistakes and incomplete accounts appeared in this book, and to appreciate criticisms and corrections in any sense.

I would like to express my deep gratitude to my advisors Professor Shi-Jian Yan and Professor Mu-Fa Chen for earnest teachings and constant helps. Professor Chen guided me to the cross research field of probability theory and Riemannian manifold, and emphasized probabilistic approaches in research, in particular, the coupling methods which he had worked on intensively. Our fruitful cooperations in this direction considerably stimulated other work included in this book. During the past decade I also greatly benefited from collaborations and communications with Professors M. Röckner, A. Thalmaier, V. I. Bogachev, F.-Z. Gong, K. D. Elworthy, M. Cranston and X.-M. Li. In particular, the work concerning the weak Poincaré inequality and applications is due to effective cooperations with Professor M. Röckner. At different stages I received helpful suggestions and encouragements from many other mathematicians, in particular, Professors S. Aida, S. Albeverio, D. Barkry, D. Chaifi, D.-Y. Chen, T. Couhlon, S. Fang, M. Fukushima, G.-L. Gong, L. Gross, E. Hsu, C.-R. Hwang, W.S. Kendall, R. Leandre, M. Ledoux, Z.-H. Li, Z.-M. Ma, P. Malliavin, Y.-H. Mao, S.-G. Peng, E. Priola, M.-P. Qian, I. Shigekawa, D. Stroock, K.-T. Sturm, Y.-L. Sun, J.-L. Wu, L. Wu, J.-A. Yan, T.-S. Zhang, Y.-H. Zhang and X.-L. Zhao. I would also like to thank Professor Yu-Hui Zhang, Mr Wei Liu and graduate students in our group for reading the draft and checking errors.

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Feng-Yu Wang
Beijing, June 2004

Contents

Chapter 0

Preliminaries

In this chapter we briefly recall some necessary preliminaries of the book from Dirichlet forms, Markov processes, spectral theory and Riemannian geometry. Results included in this part are well-known and fundamental in these fields. §0.1 and §0.2 are mainly summarized from [137], most results in §0.3 can be found in [225] and [155], and §0.4 is mainly selected from [33] and [35].

0.1 Dirichlet forms, sub-Markov semigroups and generators

Let us start with some basic facts on semigroups, resolvents and generators. Let $(\mathbb{B}, \|\cdot\|)$ be a real Banach space. A pair $(L, \mathscr{D}(L))$ is called a *linear operator* on $\mathbb{B}$ if $\mathscr{D}(L)$ is a linear subspace of $\mathbb{B}$ and $L : \mathscr{D}(L) \to \mathbb{B}$ is a linear map. We sometimes simply denote the operator by L. The operator L is called *closed* if its graph $\{(f, Lf) : f \in \mathscr{D}(L)\}$ is closed in $\mathbb{B} \times \mathbb{B}$. A linear operator $(L, \mathscr{D}(L))$ is called *closable* if the closure of its graph is the graph of a linear operator $(\bar{L}, \mathscr{D}(\bar{L}))$ which is called the *closure* of $(L, \mathscr{D}(L))$.

Definition 0.1.1 A family $\{P_t\}_{t\geqslant 0}$ of linear operators on $\mathbb{B}$ with $\mathscr{D}(P_t) = \mathbb{B}$ for all $t \geqslant 0$ is called a *strongly continuous* (or C_0-) *contraction semigroup* on $\mathbb{B}$, if

(1) $\lim_{t\to 0} P_t f = P_0 f = f, \qquad f \in \mathbb{B}.$

(2) $\|P_t\| := \sup\{\|P_t f\| : f \in \mathbb{B}, \|f\| \leqslant 1\} \leqslant 1, \qquad t \geqslant 0.$

(3) $P_t P_s = P_{t+s}, \qquad t, s \geqslant 0.$

For a given C_0-contraction semigroup $\{P_t\}_{t\geqslant 0}$ (simply denoted by P_t in the sequel), define

$$\mathscr{D}(L) := \left\{ f \in \mathbb{B} : \lim_{t\to 0} \frac{1}{t}(P_t f - f) \text{ exists in } \mathbb{B} \right\},$$

$$Lf := \lim_{t\to 0} \frac{1}{t}(P_t f - f), \qquad f \in \mathscr{D}(L).$$

Then $(L, \mathscr{D}(L))$ is a linear operator on $\mathbb{B}$, which is called the (infinitesimal) *generator* of P_t. The following well-known result provides a complete characterization for generators of C_0-contraction semigroups (see e.g. [225]).

Theorem 0.1.1 (Hille-Yoshida Theorem) *A linear operator $(L, \mathscr{D}(L))$ is the generator of a C_0-contraction semigroup if and only if*

(1) *L is densely defined, i.e. $\mathscr{D}(L)$ is dense in $\mathbb{B}$.*

(2) *For any $\lambda > 0, (\lambda - L)$ is invertible and $\|(\lambda - L)^{-1}\| \leqslant \lambda^{-1}$.*

In this case the corresponding semigroup is uniquely determined by L and is denoted by $P_t = \mathrm{e}^{tL}$, and L is closed.

Let $(L, \mathscr{D}(L))$ be the generator of a C_0-contraction semigroup P_t. We have

$$R_\lambda f := (\lambda - L)^{-1} f = \int_0^\infty \mathrm{e}^{-\lambda s} P_s f \mathrm{d}s, \qquad f \in \mathbb{B}, \quad \lambda > 0.$$

We call $\{R_\lambda : \lambda > 0\}$ the *resolvent* of L or P_t, see §0.3 for this notion of linear operators on complex Banach spaces.

Now, let us consider $\mathbb{B} := \mathbb{H}$, a real Hilbert space with inner product $\langle , \rangle$. Then an operator $(L, \mathscr{D}(L))$ provides a bilinear map $\mathscr{E} : \mathscr{D}(L) \times \mathscr{D}(L) \to \mathbb{R}$ with $\mathscr{E}(f,g) := -\langle Lf, g\rangle$ for $f, g \in \mathscr{D}(L)$. In general, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called a *bilinear form* on $\mathbb{H}$ if $\mathscr{D}(\mathscr{E})$ is a linear subspace of $\mathbb{H}$ and $\mathscr{E} : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to \mathbb{H}$ is a bilinear map. If moreover $\mathscr{E}(f, f) \geqslant 0$ for $f \in \mathscr{D}(\mathscr{E})$, then $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called a *positive definite form* on $\mathbb{H}$. For a bilinear form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, we define its symmetric part by $\tilde{\mathscr{E}}(f,g) := \frac{1}{2}(\mathscr{E}(f,g) + \mathscr{E}(g,f)), f, g \in \mathscr{D}(\mathscr{E})$. Moreover, let $\mathscr{E}_\alpha(f,g) := \alpha\langle f, g\rangle + \mathscr{E}(f,g), f, g \in \mathscr{D}(\mathscr{E}), \alpha \geqslant 0$.

Definition 0.1.2 Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a densely defined positive definite form on $\mathbb{H}$.

(1) $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called *symmetric* if $\mathscr{E}(f,g) = \mathscr{E}(g,f), \ f, g \in \mathscr{D}(\mathscr{E})$.

(2) $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called *closed* if $\mathscr{D}(\mathscr{E})$ is complete under the norm $\mathscr{E}_1^{1/2}$.

(3) $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called a *coercive closed form* on $\mathbb{H}$ if it is closed and there exists a constant $K > 0$ such that

$$|\mathscr{E}_1(f,g)| \leqslant K \mathscr{E}_1(f,f)^{1/2} \mathscr{E}_1(g,g)^{1/2}, \qquad f, g \in \mathscr{D}(\mathscr{E}). \tag{0.1.1}$$

Condition (0.1.1) is called the *weak sector condition.*

The following result gives a correspondence between the coercive closed forms and the generators of C_0-contraction semigroups.

Theorem 0.1.2 (1) *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a coercive closed form. Define*

$$\mathscr{D}(L) := \{f \in \mathscr{D}(\mathscr{E}) : \text{ the map } \mathscr{E}(f, \cdot) : \mathscr{D}(\mathscr{E}) \to \mathbb{R} \text{ is continuous under } \|\cdot\|\},$$

and for $f \in \mathscr{D}(L)$ *define* $Lf \in \mathbb{H}$ *via* $-\langle Lf, g\rangle = \mathscr{E}(f,g)$ *for all* $g \in \mathscr{D}(\mathscr{E})$. *Then* L *is the generator of a* C_0*-contraction semigroup* P_t *with resolvent* $\{R_\lambda\}_{\lambda>0}$ *satisfying*

$$R_\lambda(\mathbb{H}) \subset \mathscr{D}(\mathscr{E}) \text{ and } \mathscr{E}_\lambda(R_\lambda f, g) = \langle f, g\rangle, \qquad f \in \mathbb{H}, g \in \mathscr{D}(\mathscr{E}), \lambda > 0. \tag{0.1.2}$$

In particular, $(L, \mathscr{D}(L))$ *satisfies the weak sector condition: there exists* $K > 0$ *such that*

$$|\langle (1-L)f, g\rangle| \leqslant K\sqrt{\langle (1-L)f, f\rangle\langle (1-L)g, g\rangle}, \qquad f, g \in \mathscr{D}(L). \tag{0.1.3}$$

(2) *If* $(L, \mathscr{D}(L))$ *satisfies* (0.1.3) *and generates a* C_0*-semigroup, then there exists a unique coercive closed form* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *such that* $\mathscr{D}(\mathscr{E})$ *is the completion of* $\mathscr{D}(L)$ *with respect to* $\tilde{\mathscr{E}}_1^{1/2}$ *and*

$$\mathscr{E}(f,g) = -\langle Lf, g\rangle, \qquad f, g \in \mathscr{D}(L),$$

Furthermore, the resolvent $\{R_\lambda\}_{\lambda>0}$ *satisfies* (0.1.2).

Finally, let us consider the Markovian setting. Let $(E, \mathscr{F}, \mu)$ be a measure space and let $\mathbb{H} := L^2(\mu)$, the set of all measurable real functions which are square-integrable with respect to μ, that is, letting $\mathscr{B}(E)$ be the set of all measurable real functions on E, we have

$$L^2(\mu) := \left\{ f \in \mathscr{B}(E) : \mu(f^2) := \int_E f^2 \mathrm{d}\mu < \infty \right\}.$$

We write $f \leqslant g$ or $f < g$ if the corresponding inequality holds μ-a.e.

Definition 0.1.3 (1) A bounded linear operator $P : L^2(\mu) \to L^2(\mu)$ is called *sub-Markovian* if $0 \leqslant Pf \leqslant 1$ for all $f \in L^2(\mu)$ with $0 \leqslant f \leqslant 1$. If furthermore $P1 = 1$ then P is called a *Markov operator*. A semigroup $\{P_t\}_{t\geqslant 0}$ is called a sub-Markov (resp. Markov) semigroup if each P_t is sub-Markovian (resp. Markovian).

(2) A closed densely defined linear operator $(L, \mathscr{D}(L))$ on $L^2(\mu)$ is called a *Dirichlet operator* if $\langle Lf, (f-1)^+\rangle \leqslant 0$ for all $f \in \mathscr{D}(L)$. If moreover $1 \in \mathscr{D}(L)$ and $L1 = 0$ then L is called a *conservative Dirichlet generator*.

(3) A coercive closed form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on $L^2(\mu)$ is called a *Dirichlet form* if for any $f \in \mathscr{D}(\mathscr{E})$, one has $f^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ and

$$\mathscr{E}(f + f^+ \wedge 1, f - f^+ \wedge 1) \geqslant 0, \qquad \mathscr{E}(f - f^+ \wedge 1, f + f^+ \wedge 1) \geqslant 0. \tag{0.1.4}$$

A Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called *conservative* if $1 \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(f, 1) = \mathscr{E}(1, f) = 0$ for all $f \in \mathscr{D}(\mathscr{E})$.

Proposition 0.1.3 *(1) If $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form then so is its symmetric part $(\tilde{\mathscr{E}}, \mathscr{D}(\mathscr{E}))$.*

(2) A symmetric closed form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form if and only if for any $f \in \mathscr{D}(\mathscr{E})$ one has $f^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ and

$$\mathscr{E}(f^+ \wedge 1, f^+ \wedge 1) \leqslant \mathscr{E}(f, f). \tag{0.1.5}$$

(3) A symmetric closed form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form if and only if for any $T : \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ and $|T(x) - T(y)| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$, one has $T \circ f \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(T \circ f, T \circ f) \leqslant \mathscr{E}(f, f)$ for all $f \in \mathscr{D}(\mathscr{E})$.

(4) Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Dirichlet form. If $f \in \mathscr{D}(\mathscr{E})$ and $g \in L^2(\mu)$ satisfies $|g| \leqslant |f|, |g(x) - g(y)| \leqslant |f(x) - f(y)|$, then $g \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(g, g) \leqslant \mathscr{E}(f, f)$.

Theorem 0.1.4 *Let $(L, \mathscr{D}(L))$ generate a C_0-contraction semigroup $\{P_t\}_{t \geqslant 0}$, and let $\{R_\lambda\}_{\lambda > 0}$ be the corresponding resolvent. Then the following are equivalent.*

(1) L is a Dirichlet operator (resp. conservative Dirichlet operator).

(2) $\{P_t\}_{t \geqslant 0}$ is sub-Markovian (resp. Markovian).

(3) For each $\lambda > 0, \lambda R_\lambda$ is sub-Markovian (resp. Markovian).

If $(L, \mathscr{D}(L))$ satisfies the weak sector condition (0.1.3) and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is the associated coercive closed form, then they are also equivalent.

(4) For any $f \in \mathscr{D}(\mathscr{E})$ one has $f^+ \wedge 1 \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(f + f^+ \wedge 1, f - f^+ \wedge 1) \geqslant 0$ (resp. moreover $1 \in \mathscr{D}(\mathscr{E})$ with $\mathscr{E}(1, f) = 1$ for all $f \in \mathscr{D}(\mathscr{E})$).

Corollary 0.1.5 *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a coercive closed form associated to the generator $(L, \mathscr{D}(L))$, the semigroup $\{P_t\}_{t \geqslant 0}$ and the resolvent $\{R_\lambda\}_{\lambda > 0}$. Let P_t^* (resp. R_λ^*) be the adjoint operator (see Definition 0.3.2 below) of P_t (resp. R_λ) on $L^2(\mu)$ for $t \geqslant 0$ (resp. $\lambda > 0$), and let $(L^*, \mathscr{D}(L^*))$ be the corresponding generator. Then the following are equivalent.*

(1) $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a Dirichlet form (resp. conservative Dirichlet form).

(2) L and L^* are Dirichlet operators (resp. conservative Dirichlet operators).

(3) $\{P_t\}_{t \geqslant 0}$ and $\{P_t^*\}_{t \geqslant 0}$ are sub-Markovian (resp. Markovian).

(4) λR_λ and λR_λ^* are sub-Markovian (resp. Markovian) for each $\lambda > 0$.

In applications, $\mathscr{E}$ is often explicitly defined on a smaller domain $\mathscr{D}(\mathscr{E})$ so that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is not closed. To determine a closed form, one needs to find a closed extension of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. To this end, we introduce the notion of closability of the form.

Definition 0.1.4 A positive definite bilinear form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called *closable* if it has a closed extension $(\mathscr{E}', \mathscr{D}(\mathscr{E}'))$, i.e. $(\mathscr{E}', \mathscr{D}(\mathscr{E}'))$ is a closed form with $\mathscr{D}(\mathscr{E}') \supset \mathscr{D}(\mathscr{E})$ and $\mathscr{E}'|_{\mathscr{D}(\mathscr{E})} = \mathscr{E}$.

Proposition 0.1.6 *Let $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ be a positive definite bilinear form satisfying the weak sector condition* (0.1.1).

(1) *$(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is closable if and only if for any $\mathscr{E}$-Cauchy sequence $\{f_n\}\subset\mathscr{D}(\mathscr{E})$ (f i.e. $\mathscr{E}(f_n-f_m,f_n-f_m)\to 0$ as $n,m\to\infty$) with $f_n\to 0(n\to\infty)$ in $L^2(\mu)$, one has $\mathscr{E}(f_n,f_n)\to 0(n\to\infty)$.*

(2) *$(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is closable if and only if so is its symmetric part $(\tilde{\mathscr{E}},\mathscr{D}(\mathscr{E}))$.*

(3) *If $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is closable, then it extends uniquely to the completion of $\mathscr{D}(\mathscr{E})$ with respect to the norm $\mathscr{E}_1^{1/2}$, denoted by $(\bar{\mathscr{E}},\mathscr{D}(\bar{\mathscr{E}}))$. If moreover $\mathscr{D}(\mathscr{E})$ is dense in $L^2(\mu)$ then $(\bar{\mathscr{E}},\mathscr{D}(\bar{\mathscr{E}}))$ is the smallest coercive closed form extending $(\mathscr{E},\mathscr{D}(\mathscr{E}))$, and is called the closure of $(\mathscr{E},\mathscr{D}(\mathscr{E}))$.*

Proposition 0.1.7 (1) *Let $(L,\mathscr{D}(L))$ be a negative definite operator on $L^2(\mu)$, satisfying the weak sector condition* (0.1.3). *Define*

$$\mathscr{E}(f,g):=-\langle Lf,g\rangle,\qquad f,g\in\mathscr{D}(L).$$

Then $(\mathscr{E},\mathscr{D}(L))$ is closable on $L^2(\mu)$.

(2) *Let $(\mathscr{E}^{(k)},\mathscr{D}(\mathscr{E}^{(k)})),k\in\mathbb{N}$, be closable (resp. closed) positive definite symmetric forms on $L^2(\mu)$. Let*

$$\mathscr{D}(\mathscr{E}):=\left\{f\in\bigcap_{k\geqslant 1}\mathscr{D}(\mathscr{E}^{(k)}):\sum_{k=1}^{\infty}\mathscr{E}^{(k)}(f,f)<\infty\right\},$$

$$\mathscr{E}(f,g):=\sum_{k=1}^{\infty}\mathscr{E}^{(k)}(f,g),\qquad f,g\in\mathscr{D}(\mathscr{E}).$$

Then $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ is closable (resp. closed) on $L^2(\mu)$.

(3) *Let $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ be a coerceive closed form and $\{f_n\}\subset\mathscr{D}(\mathscr{E})$ such that $\{\mathscr{E}(f_n,f_n)\}$ is bounded and $f_n\to f\in L^2(\mu)$ as $n\to\infty$, then $f\in\mathscr{D}(\mathscr{E})$ and*

$$\lim_{n\to\infty}\tilde{\mathscr{E}}(f_n,f)=\mathscr{E}(f,f)\leqslant\varliminf_{n\to\infty}\mathscr{E}(f_n,f_n).$$

Finally, the following result (see [91, Theorem 1.5.2]) enables us to extend the domain of a Dirichlet form.

Theorem 0.1.8 *Let $(\mathscr{E},\mathscr{D}(\mathscr{E}))$ be a symmetric Dirichlet form on $L^2(\mu)$. For any measurable function f, if there exists an $\mathscr{E}$-Cauchy sequence $\{f_n\}\subset\mathscr{D}(\mathscr{E})$ such that $f_n\to f$ μ-a.e., then the limit $\mathscr{E}(f,f):=\lim_{n\to\infty}\mathscr{E}(f_n,f_n)$ exists and does not depend on the choice of $\{f_n\}$. If moreover $f\in L^2(\mu)$ then $f\in\mathscr{D}(\mathscr{E})$.*

According to Theorem 0.1.8 we may extend the Dirichlet form to the extended domain

$$\mathscr{D}_e(\mathscr{E}):=\{f\in\mathscr{B}(E):\ f_n\to f\ \mu\text{-a.e. for some}$$

$$\mathscr{E}\text{-Cauchy sequence } \{f_n\} \subset \mathscr{D}(\mathscr{E})\},$$

where $\mathscr{B}(E)$ is the set of all measurable real functions on E. Throughout the book, all real or complex functions are assumed to be finite.

0.2 Dirichlet forms and Markov processes

In this section we introduce the correspondence between Dirichlet forms and Markov processes, i.e. to show how these two objects determine each other. We first recall the notion of Markov processes.

Let E be a Hausdorff topological space with the Borel σ-field $\mathscr{F}$, that is, the σ-field induced by open sets. A stochastic process with state space E describes the behavior of a particle randomly moving on E. If the particle is allowed to move out from E, then we add a new point Δ to stand for the "died state" of the particle. Thus, the whole state space becomes $E_\Delta := E \bigcup \{\Delta\}$ equipped with the natural one-point compaction topology, i.e. a subset G of E_Δ is open if it is either an open set in E or a set containing Δ with compact complement. If in particular E itself is compact, then Δ is isolated. Let $\mathscr{F}_\Delta$ be the corresponding Borel σ-field. For any function f on E, we extend it to E_Δ by letting $f(\Delta) = 0$.

For simplicity, we only consider the standard Markov process defined on the *canonical path space* over E. A map $\omega. : [0, \infty) \to E_\Delta$ is called a canonical path if it is right continuous and has left limit at each point $t > 0$ with $\omega_t \neq \Delta$. Let Ω denote the set of all canonical pathes over E such that $\omega_t = \Delta$ for all $t \geqslant \xi(\omega) := \inf\{t \geqslant 0 : \omega_t = \Delta\}$, where ξ is called the *lifetime*. For each $t \geqslant 0$, let

$$x_t : \Omega \to E; \quad x_t(\omega) := \omega_t, \qquad \omega \in \Omega,$$

and let $\mathscr{F}_t := \sigma(x_s : s \leqslant t)$ be the smallest σ-field on Ω such that x_s is measurable for all $s \leqslant t$. Let $\mathscr{F}_\infty := \sigma(x_t : t \geqslant 0)$. The family $\{\mathscr{F}_t\}_{t \geqslant 0}$ is called the *natural filtration* of the path process over E. To make the filtration *right continuous*, let $\mathscr{F}_t^+ := \bigcap_{s>t} \mathscr{F}_s, t \geqslant 0$. In the sequel, whenever $(\Omega, \mathscr{F}_\infty)$ is equipped with a probability measure $\mathbb{P}$, the filtration under consider is automatically extended to its completion with respect to $\mathbb{P}$.

Definition 0.2.1 A family of probability measures $\{\mathbb{P}^x : x \in E_\Delta\}$ on $(\Omega, \mathscr{F}_\infty)$ is called a *Markov process* on E, if

(1) $\mathbb{P}^x(x_0 \in A) = \delta_x(A) := \mathbf{1}_A(x), x \in E_\Delta, A \in \mathscr{F}_\Delta$, where $\mathbf{1}_A$ is the indicator function of A. In particular, $\mathbb{P}^\Delta(x_t = \Delta, t \geqslant 0) = 1$.

(2) For any $\Gamma \in \mathscr{F}_\infty, \mathbb{P}^\cdot(\Gamma)$ is $\mathscr{F}_\Delta$-measurable.

(3) For any $s, t \geqslant 0$ and any $x \in E, A \in \mathscr{F}_\Delta$,

$$\mathbb{P}^x(x_{t+s} \in A|\mathscr{F}_s) = \mathbb{P}^x(x_{t+s} \in A|x_s), \qquad \mathbb{P}^x\text{-a.s.}, \tag{0.2.1}$$

where $\mathbb{P}^x(\cdot|x_s)$ is the conditional probability of $\mathbb{P}^x$ under the σ-field induced by x_s. The equation (0.2.1) is called the *Markov property* (with respect to the filtration $\{\mathscr{F}_t\}$). If for any $x \in E$ one has $\mathbb{P}^x(\xi = \infty) = 1$, then we may drop Δ and call $\{\mathbb{P}^x : x \in E\}$ a nonexplosive (or conservative) Markov process on E.

Given a Markov process $\{\mathbb{P}^x : x \in E_\Delta\}$, and given $\nu \in \mathscr{P}(E_\Delta)$, the set of all probability measures on E_Δ, let $\mathbb{P}^\nu := \int_{E_\Delta} \mathbb{P}^x \nu(\mathrm{d}x)$, which is called the distribution of the Markov process starting from ν.

In this book, we only consider the *time-homogeneous* Markov process for which the Markov property (0.2.1) can be written as

$$\mathbb{P}^x(x_{t+s} \in A|\mathscr{F}_s) = \mathbb{P}^{x_s}(x_t \in A), \qquad \mathbb{P}^x\text{-a.s.}, x \in E_\Delta, A \in \mathscr{F}_\Delta, s, t \geqslant 0. \tag{0.2.2}$$

For a time-homogenous Markov process $\{\mathbb{P}^x : x \in E_\Delta\}$, define

$$P_t f(x) := \mathbb{E}^x f(x_t) := \int_\Omega f(x_t) \mathrm{d}\mathbb{P}^x, \qquad f \in \mathscr{B}_+(E), x \in E,$$

where $\mathscr{B}_+(E)$ is the set of nonnegative measurable functions on E. Since $f(\Delta) = 0$ by convention, we have

$$P_t f(x) = \mathbb{E}^x f(x_t) 1_{\{t<\xi\}}, \qquad x \in E, t \geqslant 0.$$

It is easy to see from (0.2.2) that $\{P_t\}_{t \geqslant 0}$ is a sub-Markov semigroup on $\mathscr{B}_b(E) := \{f \in \mathscr{B}(E) : \|f\| := \sup |f| < \infty\}$, which is a Banach space and the norm is called the *uniform norm.*

To introduce the definition of strong Markov processes, let us recall the notion of a *stopping time.* A mapping $\tau : \Omega \to [0, \infty]$ is called a $\{\mathscr{F}_t\}$-stopping time if $\{\tau \leqslant t\} \in \mathscr{F}_t$ for all $t \geqslant 0$. Given a stopping time τ we define the σ-field

$$\mathscr{F}_\tau := \{\Gamma \in \mathscr{F}_\infty : \Gamma \bigcap \{\tau \leqslant t\} \in \mathscr{F}_t, \ t \geqslant 0\}.$$

Definition 0.2.2 A time-homogenous Markov process $\{\mathbb{P}^x : x \in E_\Delta\}$ is called a *strong Markov process* if for any $\{\mathscr{F}_t\}$-stopping time τ, any $\nu \in \mathscr{P}(E_\Delta)$ and any $A \in \mathscr{F}_\Delta$,

$$\mathbb{P}^\nu(x_{t+\tau} \in A|\mathscr{F}_\tau) = \mathbb{P}^{x_\tau}(x_t \in A), \qquad \mathbb{P}^\nu\text{-a.s. on } \{\tau < \infty\}. \tag{0.2.3}$$

Now, let us connect Markov processes with Dirichlet forms.